Financial Analysts Journal
Volume 68 • Number 6

A Fully Integrated Liquidity and Market Risk Model

Attilio Meucci, CFA

Going beyond the simple bid–ask spread overlay for a particular value at risk, the author introduces an innovative framework that integrates liquidity risk, funding risk, and market risk. He overlaid a whole distribution of liquidity uncertainty on future market risk scenarios and allowed the liquidity uncertainty to vary from one scenario to another, depending on the liquidation or funding policy implemented. The result is one easy-to-interpret, easy-to-implement formula for the total liquidity-plus-market-risk profit and loss distribution.

Market risk management and liquidity/funding risk management are among the top challenges in buy-side quantitative finance. Loosely speaking, market risk is the uncertainty of the profit and loss (P&L) at a given investment horizon in the future; liquidity risk is the potential loss, arising from the action of trading, with respect to a reference mark-to-market value.

The literature on market risk management, market risk estimation, and estimation error is enormous (for a review, see Meucci 2005). Liquidity risk has been addressed in a variety of contexts and under various names in the financial literature (for a review, see Hibbert, Kirchner, Kretzschmar, Li, McNeil, and Stark 2009), including these studies: measures of liquidity (Amihud 2002); axiomatic definition of liquidity impact (see, e.g., Çetin, Jarrow, and Protter 2004; Acerbi and Scandolo 2007); optimal execution (see, e.g., Bertsimas and Lo 1998; Almgren and Chriss 2000; Obizhaeva and Wang 2005; Gatheral, Schied, and Slynko 2012); transaction cost–aware portfolio optimization (see, e.g., Lobo, Fazel, and Boyd 2007; Lo, Petrov, and Wierzbicki 2003; Engle and Ferstenberg 2007).

In this article, I propose a new methodology to integrate market risk, liquidity risk, and funding risk for all asset classes. Using this approach, I overlaid liquidity/funding uncertainty on market risk scenarios at the future investment horizon. The distribution of the liquidity uncertainty depends on the amount liquidated at the future horizon, which, in turn, depends on the specific market scenario.

Attilio Meucci, CFA, is head of portfolio construction at Kepos Capital, New York City.

Discussion of findings. My main result is the liquidity-plus-market-risk P&L distribution formula (Equation 10 herein), which is easy to interpret and easy to implement. Other approaches to modeling market and liquidity risk jointly have been explored (see, e.g., Bangia, Diebold, Schuermann, and Stroughair 2002; Jorion 2007). My methodology improves on the current methodologies in seven ways:

- My liquidity model goes beyond a deterministic bid–ask spread overlay to a pure market risk component. Indeed, it models the full impact of any actual liquidation schedule, including impact uncertainty and impact correlations, as well as the differential impact between trading quickly and trading slowly.
- My liquidity model is state dependent; for instance, in those scenarios where the market is down and volatile, the adverse impact of any liquidation schedule is worse and, therefore, so is the liquidity of the portfolio.
- My liquidity model addresses both exogenous liquidity risk (arising from market conditions beyond our control) and funding risk (i.e., endogenous liquidity risk): Using this framework, one can model more aggressive liquidation schedules on capital-intensive securities specifically in those market scenarios that give rise to very negative P&L, all while no liquidation occurs in positive P&L scenarios.
- My liquidity model includes all the features of the market risk component beyond mean and variance. In particular, it models the P&L of not only nonsymmetrical tail events but also such nonlinear securities as complex derivatives.

- My liquidity model explicitly addresses the issue of estimation error, allowing for fast distributional stress testing via the fully flexible probabilities methodology (discussed later in the article).
- My methodology allows for a novel decomposition of risk into a market risk component and a liquidity risk component.
- My methodology also allows for a natural definition of the portfolio's liquidity score in monetary units.

The General Framework

To build my liquidity-plus-market-risk model, we must follow four steps (see **Figure 1** for a schematic depiction):

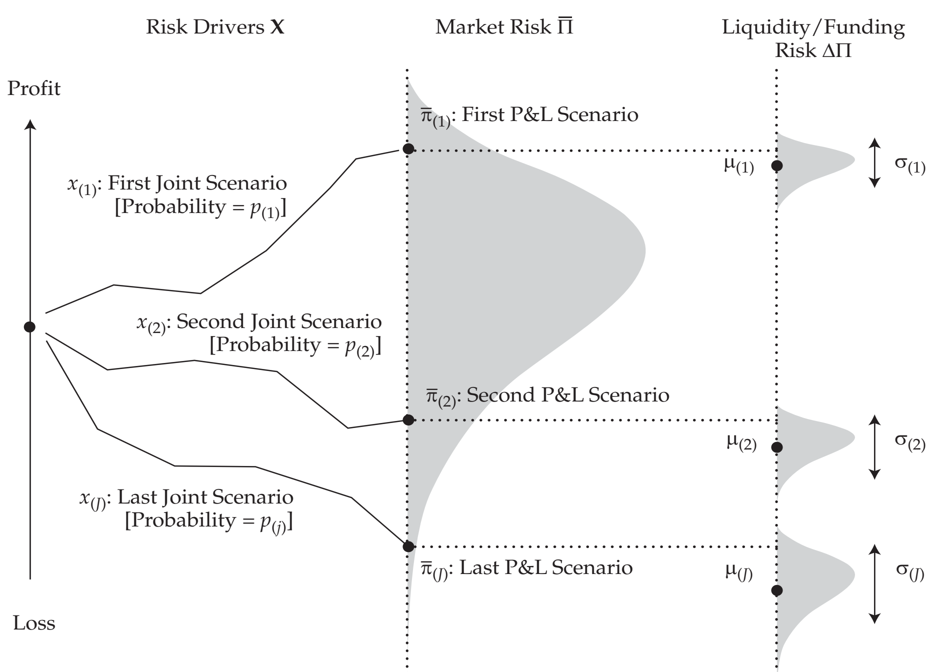
1. We first model the pure market risk component of the P&L from now to the investment horizon via precise scenario repricing and the fully flexible probabilities methodology.
2. We then overlay on the pure market risk the impact of the liquidation schedules at the horizon, which can be different in different scenarios. This way, we can model both exogenous liquidity risk and endogenous liquidity risk (i.e., funding risk).
3. Next, we aggregate all the market scenarios and their liquidity adjustments into one total liquidity- and funding-adjusted P&L distribution.
4. From the total liquidity-adjusted P&L distribution, we compute all the summary statistics, including standard deviation, value at risk (VaR), and conditional value at risk (CVaR). We then decompose these statistics into the market risk contribution and the liquidity risk contribution via a novel, explicit formula and compute a novel liquidity score in monetary terms.

Market Risk: Fully Flexible Probabilities. Let us first consider a general market of N securities. We denote with $\bar{\Pi}_n$ the mark-to-market P&L delivered by one unit of the generic nth security between the current time and the future investment horizon of the portfolio manager. The security unit is one share for stocks, one contract for futures, a given reference notional for swaps, and so on. The investment horizon varies widely across portfolio managers, although it is typically on the order of one day.

Across all asset classes in general, the stochastic behavior of the projected P&L $\bar{\Pi}_n$ is fully determined by the evolution of D risk drivers $\mathbf{X} \equiv (X_1, \ldots, X_D)'$. Thus, the P&L of the generic nth security is a deterministic function π_n of the risk drivers:

$$\bar{\Pi}_n = \pi_n(\mathbf{X}), \tag{1}$$

Figure 1. The Overlay on Market Risk of Liquidity/Funding Risk Execution Price Uncertainty at the Investment Horizon



where $n = 1, \ldots, N$. This framework can be applied to all asset classes (see Meucci 2011b, 2011c). For instance, for a stock, the risk driver is its log price at the horizon $X_n = \ln S_{n,t}$ and the mark-to-market P&L is $\bar{\Pi}_n = e^{X_n} - s_{n,0}$. For an option, **X** represents the underlying security and all the entries of the implied volatility surface at the investment horizon; the pricing function π_n can be specified exactly in terms of the Black–Scholes formula, or it can be approximated by a Taylor expansion whose coefficients are the well-known Greeks: the deltas, vegas, gammas, vannas, volgas, and so on.

Let us now consider a portfolio of N holdings $\mathbf{h} \equiv (h_1, \ldots, h_N)'$. Here, *holdings* means the number of units of the given securities (i.e., the number of shares for stocks, the number of contracts for futures, etc.); it does not mean portfolio weights, which might not be properly defined for long–short positions or for such leveraged instruments as swaps and futures (see Meucci 2010b).

For portfolio **h**, the mark-to-market P&L is the sum of the contributions from each position—that is, $\bar{\Pi} = \sum_n h_n \pi_n(\mathbf{X})$.

A flexible approach to modeling market risk (i.e., the distribution of the risk drivers **X**) is the fully flexible probabilities framework in Meucci (2010a); again, see Figure 1. In this approach, the distribution of **X** is modeled by two sets of variables—a set of joint scenarios $j = 1, \ldots, J$ for the risk drivers $\mathbf{x}_{(j)} \equiv (x_{1,(j)}, \ldots, x_{D,(j)})'$, which can be generated historically or via Monte Carlo simulation, and their respective relative probabilities $p_{(j)}$ (one probability for each joint scenario):

$$\mathbf{X} \sim \left[\mathbf{x}_{(j)}, p_{(j)}\right]_{j=1,\ldots,J}. \tag{2}$$

The fully flexible probabilities framework (Equation 2) is computationally efficient because the distribution of the portfolio P&L follows immediately by computing the portfolio P&L in each scenario, all while the relative probabilities remain unchanged:

$$\bar{\Pi} \sim \left[\bar{\pi}_{(j)}, p_{(j)}\right]_{j=1,\ldots,J}, \tag{3}$$

where each P&L scenario is obtained by applying the pricing function (Equation 1) to the respective scenario for the risk drivers

$$\bar{\pi}_{(j)} = \sum_n h_n \pi_n\left(\mathbf{x}_{(j)}\right). \tag{4}$$

One special case of the fully flexible probabilities framework (Equation 2) is when all the probabilities are equal—$p_{(j)} = 1/J$—which is typical in the so-called historical simulations approach.

One significant benefit of the fully flexible probabilities framework is that it allows us to model all sorts of non-normal market distributions, as well as such nonlinear instruments as close-to-expiry options.

Furthermore, in the fully flexible probabilities framework, we can stress test the probabilities $p_{(j)}$ to emphasize different periods or different market conditions by using a variety of advanced techniques: exponential smoothing, kernels, entropy pooling, and so on. Thus, we can perform all sorts of stress tests on the P&L distribution, as we will see later in the article.

Finally, with the fully flexible probabilities approach, we can separate the computationally heavy part of the process—namely, the computation of the P&L of each security $\pi_n(\mathbf{x}_{(j)})$—from the computationally light part, namely, the sum that yields the portfolio P&L scenarios $\bar{\pi}_{(j)}$ in Equation 4 and the stress testing of the market distribution via $p_{(j)}$. The heavy part can be run overnight using a batch process, whereas the light part can be performed on the fly for portfolios of thousands of securities. For more details on the fully flexible probabilities framework, see Meucci (2010a).

Liquidity Risk: Market Impact over Liquidation Horizon. In standard risk and portfolio management applications, the P&L function (Equation 1) yields the distribution of the P&L. However, for a specific realization **x** of the risk drivers **X** at the investment horizon, an uncertainty $\Delta\Pi_n$ in the P&L is generated by the generic nth position because of liquidity-related issues (see Figure 1). This liquidity adjustment, $\Delta\Pi_n$, is determined by three factors.

The first factor affecting $\Delta\Pi_n$ is the state of the market, which can be included among the risk drivers **X**. For example, when the CBOE Volatility Index spikes, liquidity can suddenly decrease.

The second factor affecting the liquidity adjustment, $\Delta\Pi_n$, is the amount liquidated at the future investment horizon. Let us denote with $\Delta\mathbf{h} \equiv (\Delta h_1, \ldots, \Delta h_N)'$ this action, or liquidation schedule. For mark-to-market purposes, we do not liquidate any position (i.e., $\Delta\mathbf{h} = \mathbf{0}$), and thus there is no impact on the P&L, $\Delta\Pi_n = 0$. At the other extreme, in a full liquidation of a large portfolio (i.e., $\Delta\mathbf{h} = -\mathbf{h}$), we generate an impact on the prices and, therefore, on the P&L. The price impact has two components: (1) a permanent component, which must be linear and thus cannot contribute to the cost of a round-trip trade because otherwise there would be arbitrage opportunities (see Gatheral 2010), and (2) a temporary component, which we expect will adversely affect the portfolio, $E\{\Delta\Pi_n\} < 0$ (see Figure 1).

The third factor affecting the liquidity adjustment, $\Delta\Pi_n$, is the execution horizons $\tau \equiv (\tau_1, \ldots, \tau_N)'$ for the liquidations $\Delta\mathbf{h} \equiv (h_1, \ldots, h_N)'$. As in Almgren and Chriss (2000), longer execution horizons generate less impact on prices but more uncertainty; shorter execution horizons generate more impact but less uncertainty.

For now, let us model the liquidity adjustment for the P&L of each position as a normal distribution, where both mean and volatility depend on the three factors:

$$\Delta\Pi_n \sim N\left(\mu_n, \sigma_n^2\right). \tag{5}$$

For those readers concerned that the normal assumption for the liquidity adjustments (Equation 5) might not be realistic, I show in Appendix A—available from the author upon request—how to generalize the normal assumption to more realistic skewed and thicker-tailed distributions. In Appendix A, I also derive explicit expressions for μ_n, which is negative, and σ_n (in Equation 5) across asset classes for a broad range of liquidation strategies $\Delta\mathbf{h}$ and arbitrary execution time frames τ. These expressions, which represent one of the innovations in my methodology, draw on four sets of previous results: (1) the universal square root impact form, justified empirically and theoretically in Grinold and Kahn (1999); Almgren, Thum, Hauptmann, and Li (2005); and Toth, Lemperiere, Deremble, De Lataillade, Kockelkoren, and Bouchaud (2011); (2) the moments of a general impact function in Almgren (2003); (3) the optimal execution paradigm in Almgren and Chriss (2000); and (4) the equivalence between volatility and market activity in Ané and Geman (2000).

Here, I report only the simplest of such expressions, namely, those that follow from the volume-weighted average price (VWAP) execution.

The VWAP-derived mean of the liquidity adjustment (Equation 5) is given by

$$\mu_n = -\alpha_n(\mathbf{x})e_n|\Delta h_n| - \beta_n(\mathbf{x})e_n\bar{\sigma}_n\frac{|\Delta h_n|^{3/2}}{\sqrt{v_n}}, \tag{6}$$

where

α_n = an estimate of the commissions plus half the bid–ask spread as a percentage of the exposure e_n of one unit of the nth security (e.g., for stocks, e_n is the price of one share), which can vary with the liquidity and market conditions $\mathbf{x}$

β_n = a coefficient approximately constant across securities within the same asset class but which can vary with the liquidity and market conditions $\mathbf{x}$

$\bar{\sigma}_n$ = an estimate of the average annualized P&L volatility of one unit of the nth security as a percentage of the exposure e_n

v_n = an estimate of the total number of units of the nth security traded by the market over the execution period τ_n

The VWAP-derived uncertainty of the liquidity adjustment (Equation 5) is given by

$$\sigma_n = -\delta_n(\mathbf{x})\sqrt{v_n}e_n|\Delta h_n|, \tag{7}$$

where δ_n is a coefficient that varies with the market conditions $\mathbf{x}$ and has a dimension inverse to $\sqrt{v_n}$ and the remaining quantities are as defined in Equation 6.

Note that when the execution period τ_n of a given liquidation Δh_n increases, so does the cumulative trading activity v_n over the period. Thus, as intuition suggests, the expected impact (Equation 6) with a longer execution period τ_n decreases whereas the impact uncertainty (Equation 7) increases. Note also how Equations 6 and 7 make sense from a dimensional perspective. For example, if a 2-for-1 stock split occurs, α_n, β_n, and $\bar{\sigma}_n$ (in Equation 6) do not change, e_n halves, $|\Delta h_n|$ doubles, v_n doubles, and thus the expected liquidity adjustment in money terms μ_n does not change.

The correlations $\rho_{n,m}$ among the liquidity adjustments (Equation 5) for two positions are similar to the respective market risk correlations $\hat{\rho}_{n,m}$ but are empirically likely to be closer to the maximum value of 1 because liquidity risk is less diversifiable than market risk. This phenomenon can be modeled with a common shrinkage parameter γ close to 1, as follows:

$$\rho_{n,m} = \gamma + (1-\gamma)\hat{\rho}_{n,m}. \tag{8}$$

The model parameters α_n, β_n, δ_n, and γ must be calibrated for each traded market, say, stocks, foreign exchanges, and so on (discussed later in the article).

Once we have computed the liquidity adjustments (Equation 5) for all the positions and correlations (Equation 8), we can aggregate all the adjustments $\Delta\Pi = \Sigma_n\Delta\Pi_n$ and compute the distribution of the liquidity adjustment $\Delta\Pi$ for the whole portfolio:

$$\Delta\Pi \sim N\left(\mu, \sigma^2\right), \tag{9}$$

where $\mu = \Sigma_n\mu_n$ and $\sigma^2 = \Sigma_{n,m}\sigma_n\sigma_m\rho_{n,m}$.

Total and Funding Risk. The total portfolio P&L $\Pi = \bar{\Pi} + \Delta\Pi$ is the sum of the pure mark-to-market component $\bar{\Pi}$ (Equation 3) and the liquidity adjustment $\Delta\Pi$ (Equation 9). As shown in Appendix A (available from the author upon request), the probability density for any generic value y of the portfolio P&L is the following simple formula, obtained via a conditional convolution:

$$f_{\Pi}(y) = \sum_j \frac{p_{(j)}}{\sigma_{(j)}} \varphi\left[\frac{y - \bar{\pi}_{(j)} - \mu_{(j)}}{\sigma_{(j)}}\right], \quad (10)$$

where

φ = the standard normal distribution density (this assumption can be easily generalized to more realistic skewed and thicker-tailed distributions; see Appendix A)

$\bar{\pi}_{(j)}$ = the pure market risk P&L in the generic jth scenario (Equation 4)

$p_{(j)}$ = the respective probability according to the fully flexible probabilities framework (Equation 2)

$[\mu_{(j)}, \sigma_{(j)}]$ = the portfolio liquidity adjustment parameters (μ, σ) that appear in Equation 9, evaluated in the generic jth scenario

In particular, in the limit of no liquidity adjustment (i.e., $\mu_{(j)} \approx 0$ and $\sigma_{(j)} \approx 0$) and when all the probabilities are equal (i.e., $p_{(j)} = 1/J$), the distribution (Equation 10) becomes the standard empirical distribution of the mark-to-market P&L scenarios.

To the best of my knowledge, the simple yet general and flexible total liquidity-plus-market-risk portfolio P&L distribution (Equation 10) is new and represents one of the main contributions of my methodology. As one can verify in the MATLAB code available at www.symmys.com/node/350, we can compute the total P&L distribution (Equation 10) with thousands of positions N and thousands of scenarios J in seconds.

Note how we can make the liquidation schedule of each position depend on the market scenario $\Delta h_n \mapsto \Delta h_{n,(j)}$ with no additional computational cost. The same holds for the execution periods of the liquidation schedule $\tau_n \mapsto \tau_{n,(j)}$. Hence, in our framework, we can easily model stochastic liquidations (see Brigo and Nordio 2010).

As a result, we can measure and stress test funding risk: In a scenario j where the mark-to-market P&L $\bar{\pi}_{(j)}$ is very negative, the company will liquidate faster, via short execution schedules $\tau_{n,(j)}$, and in larger amounts, via the transaction $\Delta h_{n,(j)}$, those securities n with the smallest marginal cost of liquidation per unit of capital (i.e., the most liquid). For instance, during the recent financial crisis, many hedge funds faced with funding issues chose to liquidate quant-equity strategies.

Summary Statistics and Liquidity Score. With the distribution (Equation 10) of the total liquidity-plus-market-risk portfolio P&L $\Pi = \bar{\Pi} + \Delta\Pi$, we can compute all sorts of risk statistics. Standard statistics include expected return, standard deviation, Sharpe ratio, and VaR and CVaR for any tail risk level. More advanced statistics include expected utility and certainty equivalent for any kind of utility function and measures of satisfaction for arbitrary risk-aversion spectra (for a review, see Meucci 2005).

Because the total P&L $\Pi = \bar{\Pi} + \Delta\Pi$ is simply the sum of the mark-to-market component and the liquidity component, it is possible, in principle, to decompose all the standard risk measures, including standard deviation, VaR, and CVaR, into so-called marginal contributions (see Hallerbach 2003; Gourieroux, Laurent, and Scaillet 2000; Tasche 2002). For example, we obtain the following decomposition for the CVaR:

$$CVaR\{\Pi\} = \underbrace{\partial_{\bar{\Pi}} CVaR\{\Pi\}}_{\text{Market risk}} + \underbrace{\partial_{\Delta\Pi} CVaR\{\Pi\}}_{\text{Liquidity risk}}. \quad (11)$$

In practice, the total CVaR on the left-hand side follows from the numerical computation of the total P&L probability density function (Equation 10). The liquidity risk contribution is provided by the following formula (which I believe is a new result):

$$\partial_{\Delta\Pi} CVaR\{\Pi\} = \sum_j p_{(j)} \frac{\mu_{(j)} - \sigma_{(j)}\varphi\left(z_{(j)}\right)}{\Phi\left(z_{(j)}\right)}, \quad (12)$$

where $z_{(j)} \equiv \left(F_{\Pi}^{-1}(\alpha) - \bar{\pi}_{(j)} - \mu_{(j)}\right) / \sigma_{(j)}$, α is the CVaR confidence, and Φ is the standard normal cumulative distribution function. Then, the market risk contribution to $\partial_{\bar{\Pi}} CVaR\{\Pi\}$ follows as the difference between the CVaR and its liquidity contribution (see Appendix A for more details).

In addition, we can measure the liquidity score for the portfolio. A portfolio is liquid when the liquidity adjustment $\Delta\Pi$ plays a minor role in the total portfolio P&L Π. Because the liquidity adjustment, on average, moves the P&L downward, it is natural to define a liquidity score as the percentage deterioration in the left tail, as measured by a standard metric (e.g., the CVaR with 90% confidence). Accordingly, we define the liquidity score (LS) of a portfolio as the difference between the pure market risk $CVaR\{\bar{\Pi}\}$ and the total liquidity-adjusted risk $CVaR\{\Pi\}$, normalized as a return:

$$\text{LS} = \frac{CVaR\{\bar{\Pi}\}}{CVaR\{\Pi\}}. \quad (13)$$

The liquidity score is always greater than 0 and less than 1. When the impact of liquidity is negligible (i.e., $\Pi = \bar{\Pi}$), the liquidity score approaches the upper boundary of 1. When the impact of liquidity is substantial (i.e., the left tail of Π is much more negative than the left tail of $\bar{\Pi}$), the liquidity score approaches the lower boundary of zero.

Case Study: Liquidity Management for Equity Portfolios

Let us now consider a case study with portfolios of stocks in the S&P 500 Index. For all the details, see the documented MATLAB code at www.symmys.com/node/350.

The Setup. In our chosen market, there are N = 500 securities. We choose an investment horizon t = one day. The risk drivers are the horizon log prices of each stock $X_n \equiv \ln S_n$, and thus the P&L pricing function (Equation 1) becomes

$$\bar{\Pi}_n = e^{X_n} - s_n, \tag{14}$$

where the uppercase letters denote future random variables and the lowercase letters denote currently known numbers.

We also include an additional risk driver X_0 that summarizes the overall level of liquidity in the market. In particular, we use the Morgan Stanley liquidity factor, which is the cumulative P&L of a dynamic portfolio rebalanced to stay long liquid stocks and short illiquid stocks. Thus, in our case study, we have $D = 1 + N = 501$ risk drivers. We collect daily data over a period of approximately four years for the risk drivers and other variables, such as traded volumes.

We model the distribution of the risk drivers $\mathbf{X} \equiv (X_0, X_1, \ldots, X_N)'$ via the historical distribution of their day-to-day changes. With our database of about four years of daily observations, we obtain $J \approx 1{,}000$ scenarios. To start, we assign these historical scenarios equal probability weights $p_{(j)} = 1/J$. With these scenarios and probabilities, we specify the fully flexible probabilities framework that defines market risk (Equation 2).

Next, for each stock, we calibrate the parameters α_n, β_n, and δ_n for the liquidity adjustment (Equations 6 and 7). We calibrate these parameters as functions $\alpha_n(x_0)$, $\beta_n(x_0)$, and $\delta_n(x_0)$ of the liquidity index x_0 because different liquidity regimes in the market correspond to different market impact parameters. We also calibrate the correlation shrinkage parameter for the liquidity correlations in Equation 8 as $\gamma = 0.90$.

We are now ready to start our analysis.

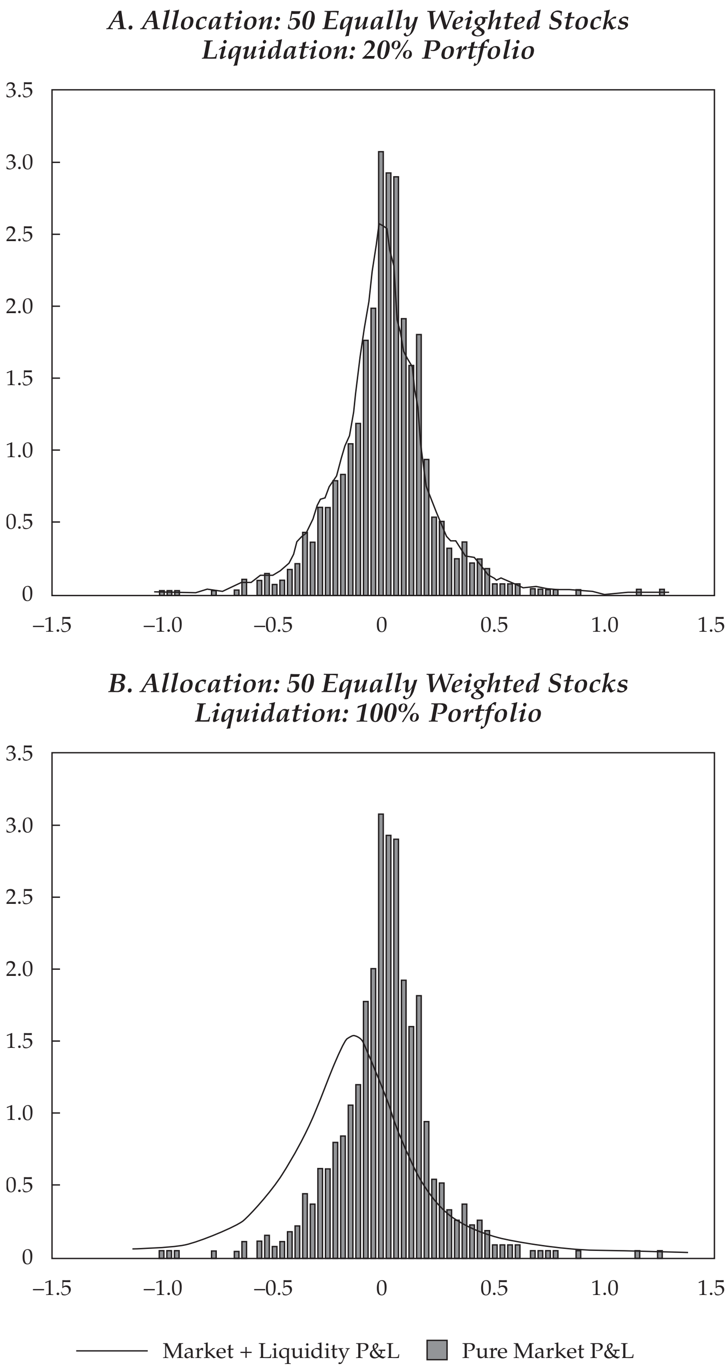
Example 1: Base-Case Equally Weighted Portfolio. Let us consider an equally weighted portfolio of 50 stocks $h_1 s_1 = \ldots h_{50} s_{50}$ whose total notional $h_1 s_1 + \ldots + h_{50} s_{50}$ amounts to 30% of the average daily volume of the S&P 500. The ensuing pure market risk distribution (Equation 3) for the P&L $\bar{\Pi}$ in our portfolio is represented by the histograms in **Figure 2**. We then stress test a proportional liquidation of 20% of all the assets (i.e., $\Delta h_n = -0.2 h_n$). We also consider the same execution period of τ_n = one day for all the assets (i.e., all trades are executed between the investment horizon, which is tomorrow's close, and the close one day thereafter). The distribution line in Panel A of Figure 2 represents the distribution (Equation 10) for the liquidity-plus-market-risk P&L $\Pi = \bar{\Pi} + \Delta\Pi$, in the case of our portfolio. Note how this liquidation can be absorbed by the market with relatively little impact, which is reflected in the liquidity score (Equation 13): LS ≈ 96%.

Example 2: Aggressive Liquidation. We now consider the previous setting but with an aggressive full liquidation (i.e., $\Delta h_n = -h_n$). The distribution of the total P&L $\Pi = \bar{\Pi} + \Delta\Pi$ becomes the line in Panel B of Figure 2. Note how the liquidity impact, now more invasive, shifts and twists the P&L distribution toward the left tail. Indeed, LS ≈ 71%.

Example 3: Liquidity Diversification. Let us now look at diversification issues by performing the full liquidation on a heavily concentrated portfolio of one stock and on a mildly concentrated equally weighted portfolio of five stocks, both with the same initial capital as in the previous setting. The results are depicted in Panels A and B of **Figure 3**, with LS ≈ 66% and LS ≈ 69%, respectively. Note how market risk (i.e., the width of the histogram) shrinks as we progress from 1 to 5 to 50 stocks. Liquidity, however, is not easily diversifiable.

Example 4: Funding Risk with Different Trades in Different Market Scenarios. With respect to funding risk, we can consider our original 50-stock portfolio but with a larger total value of assets under management, equal to the average daily volume of the S&P 500. The one-day full liquidation schedule that we previously tested is unrealistic. Instead, we intend to free up cash as needed, in adverse market scenarios only. More precisely, let us suppose that in a given scenario j, the market P&L $\bar{\pi}_{(j)}$ falls below a critical negative threshold $\bar{\pi}$, which we set as a fraction of the market P&L volatility. In this scenario, we will need to free up the amount of cash $\bar{\pi} - \bar{\pi}_{(j)}$. Let us denote with m_n the dollar margin to invest, long or short, in one share of the nth stock.

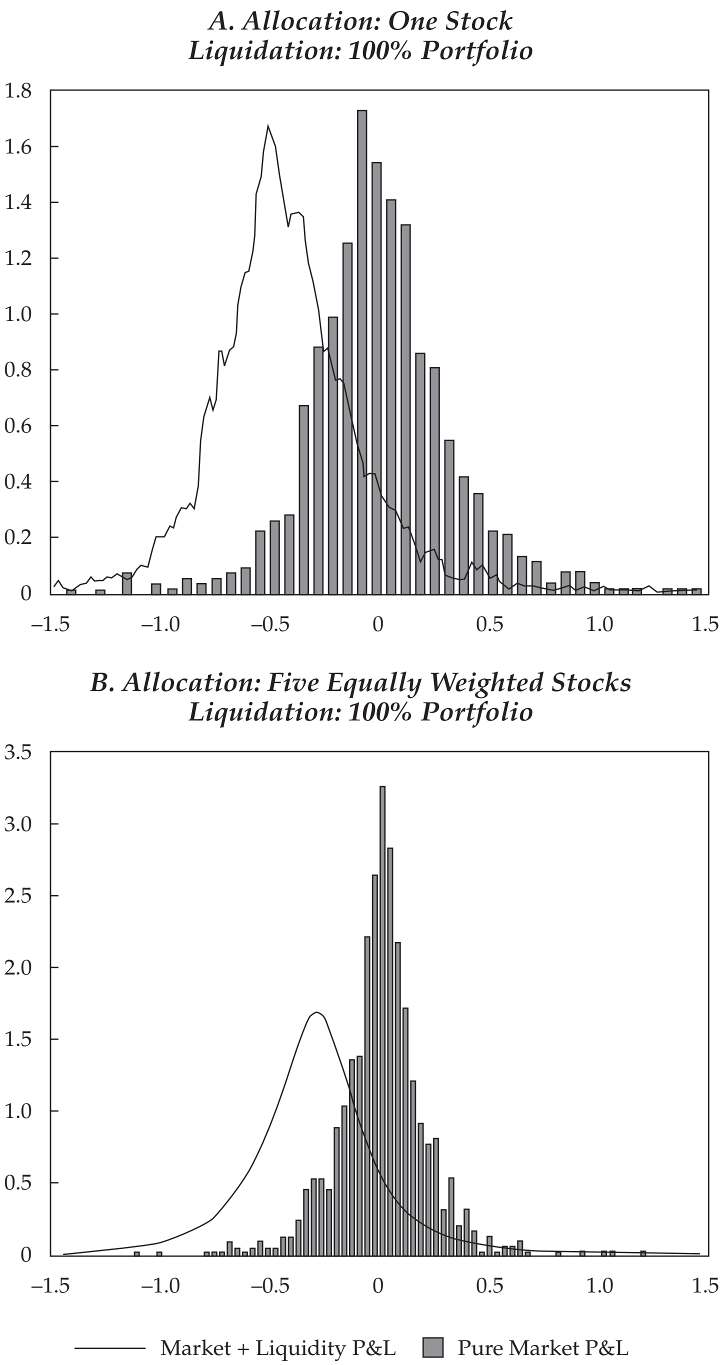
Figure 2. Adverse Impact of Liquidity Risk on the Portfolio



Then, the liquidation of a fraction θ of the current exposure $\Delta h_n = -\theta h_n$ makes available the amount of cash $\theta|h_n|m_n$. Therefore, to cover our cash needs, we liquidate a scenario-dependent portion $\theta_{(j)}$ of our portfolio, which is zero if the P&L $\overline{\pi}_{(j)}$ is above the threshold; otherwise, it is determined by the relationship $\sum_n \theta_{(j)}|h_n|m_n = \overline{\pi} - \overline{\pi}_{(j)}$. The results are shown in Panel A of **Figure 4**. Note how funding risk increasingly hits the far left tail of the P&L. Thus, despite the overall relatively small liquidity perturbation, LS ≈ 84%.

Example 5: Funding Risk with Different Trades and Execution Periods in Different Market Scenarios. Let us now delve deeper into funding risk. We need not only scenario-dependent liquidation policies but also scenario-dependent

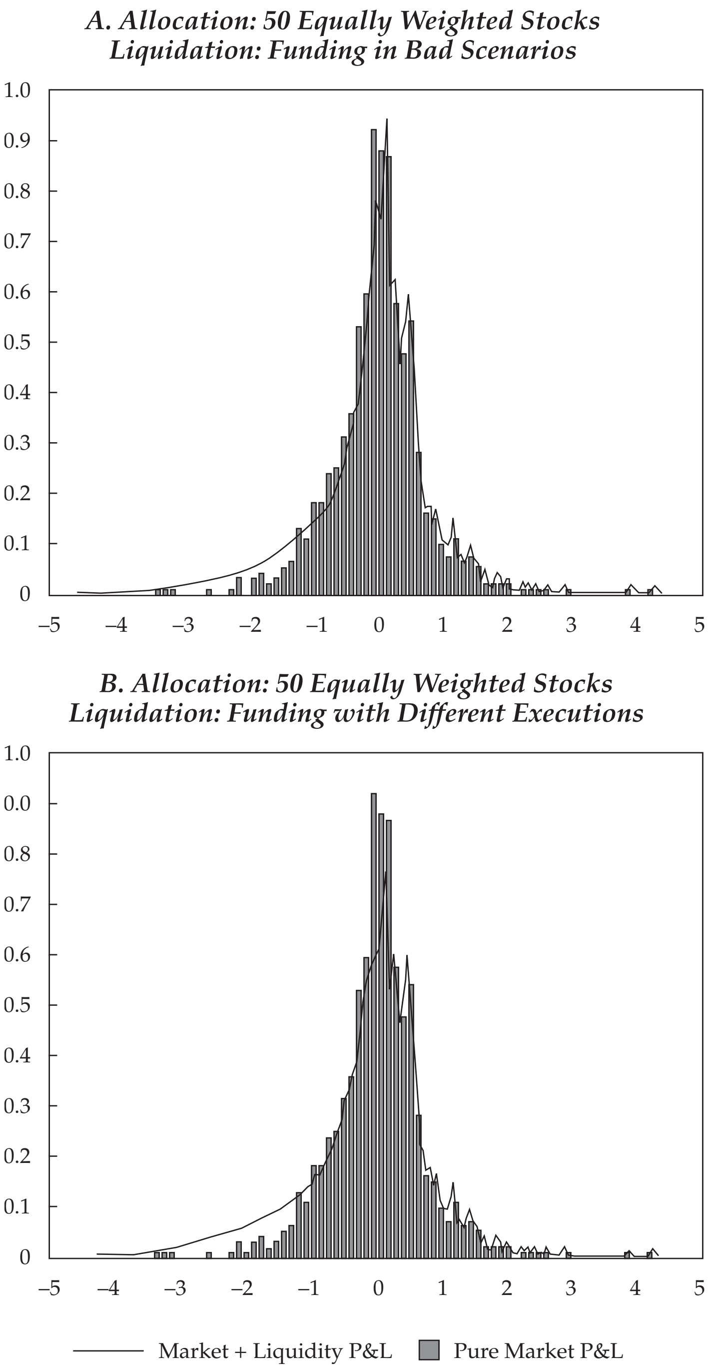
Figure 3. Diversifiability of Market Risk and Liquidity Risk



execution schedules. In very negative market P&L scenarios $\bar{\pi}_{(j)}$, far below the critical threshold $\bar{\pi}$, the need for cash is more immediate than in milder scenarios, and thus all the execution schedules $\tau_{n,(j)}$ occur faster, creating a vicious circle of heightened liquidity risk. In this scenario, LS ≈ 78%. The P&L distribution is displayed in Panel B of Figure 4. Note how funding risk is further exacerbated in Panel A of Figure 4, where all the execution periods equal one day (i.e., $\tau_{n,(j)} = 1$) for all stocks n and all scenarios j.

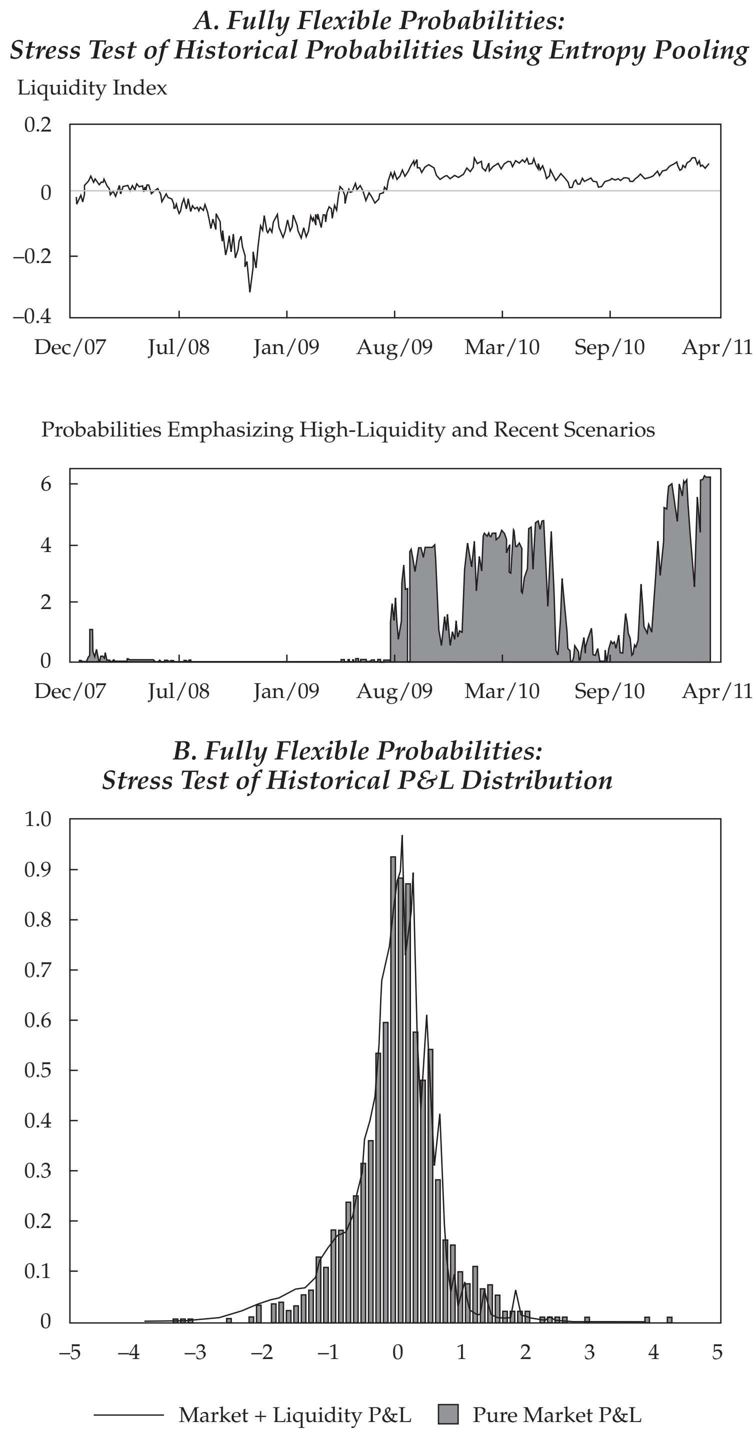
Example 6: Stress Testing Liquidity Risk and Market Risk. Let us now leverage the fully flexible probabilities framework to address the issue of estimation risk. Any market distribution estimated from past historical observations is never equal

Figure 4. Funding Risk: State-Dependent Liquidation Schedules



to the true, unknown future market distribution. Hence, we need to stress test different estimates. So far, all the historical scenarios $j = 1, \ldots, J$ in the total portfolio distribution (Equation 10) have been given equal weight by setting all the probabilities of the scenarios to be equal (i.e., $p_{(j)} = 1/J$). With the fully flexible probabilities framework, we can modify the relative weights $p_{(j)}$ of the scenarios to better reflect the current state of the market. More precisely, we can observe in the time series of the Morgan Stanley Liquidity Factor, displayed in the top portion of Panel A in **Figure 5**, that the market has recently been relatively liquid. Accordingly, we give more probability weight to recent scenarios, as well as to past scenarios in which the liquidity index was high. The technique we use to perform this blending in the fully flexible probabilities framework is called entropy pooling (for details,

Figure 5. Fully Flexible Probabilities Stress Test of Market Risk



see Meucci 2011a). The outcome is the nonequal probabilities $p_{(j)}$ shown in the bottom portion of Panel A in Figure 5.

We can then use these probabilities to recompute the total liquidity-plus-market-risk portfolio P&L distribution (Equation 10) in the same portfolio with funding-driven liquidation schedules, as in Example 5. Note that the fully flexible probabilities stress test comes at no additional cost: To obtain any new estimates of the P&L distribution, we simply have to plug the new probabilities $p_{(j)}$ into the total distribution formula (Equation 10). Panel B of Figure 5 depicts the results. As expected, the liquidity score has risen, to around 93%.

Conclusion

I have presented a framework to model, measure, and act on market risk, liquidity risk, and funding risk across all financial instruments and trading styles, and I have demonstrated my framework in practice in a case study.

My approach goes beyond the simple bid–ask spread adjustment to a VaR number because it models the full random impact of any liquidation schedule on prices and, therefore, on the portfolio P&L. Furthermore, in my framework, liquidity depends on the state of the market at the future investment horizon.

My approach culminates in a simple, new formula (Equation 10) for the liquidity-plus-market-risk distribution of the portfolio P&L, which can be computed in seconds for portfolios with thousands of securities.

The liquidity-plus-market-risk P&L formula (Equation 10) allows us to dissect, stress test, and eventually act on all the components of risk.

By changing the probabilities $p_{(j)}$ according to the fully flexible probabilities framework, we can explore the effects of various market environments on the liquidity-adjusted P&L distribution, such as low/high liquidity, low/high volatility, and so on.

By modifying the liquidation stress test Δh_n of each security $n = 1, \ldots, N$ and the respective execution periods τ_n, we can obtain different liquidity adjustment parameters $\mu_{(j)}$ and $\sigma_{(j)}$ in each future market scenario j.

By making the liquidation actions $\Delta h_{n,(j)}$ and their respective execution periods $\tau_{n,(j)}$ depend on the market scenarios j, we can stress test more flexible and realistic situations, such as funding-driven massive and fast liquidations only when the portfolio incurs large losses.

Once we have computed the liquidity-plus-market-risk P&L distribution (Equation 10), we can extract all sorts of risk indicators, such as standard deviation, VaR, CVaR, and the newly introduced liquidity score (Equation 13), which is the tail return of the liquidity-adjusted portfolio over the tail return of the same portfolio absent any liquidity issues.

Finally, we can compute explicitly the contributions to standard deviation, VaR, and CVaR from liquidity risk and market risk as in Equations 11 and 12, another original contribution of my new methodology.

The author is grateful to Carlo Acerbi, Rob Almgren, Garli Beibi, Damiano Brigo, Chuck Crow, Gianluca Fusai, Jim Gatheral, Winfried Hallerbach, Arlen Khodadadi, Bob Litterman, Marcos Lopez De Prado, Stefano Pasquali, and Luca Spampinato.

This article qualifies for 1 CE credit.

References

Acerbi, C., and G. Scandolo. 2007. "Liquidity Risk Theory and Coherent Measures of Risk." Working paper (November).

Almgren, R. 2003. "Optimal Execution with Nonlinear Impact Functions and Trading-Enhanced Risk." *Applied Mathematical Finance*, vol. 10, no. 1:1–18.

Almgren, R., and N. Chriss. 2000. "Optimal Execution of Portfolio Transactions." *Journal of Risk*, vol. 3, no. 2:5–39.

Almgren, R., C. Thum, E. Hauptmann, and H. Li. 2005. "Equity Market Impact." *Risk Magazine*, vol. 18 (1 July):57–62.

Amihud, Y. 2002. "Illiquidity and Stock Returns: Cross-Section and Time-Series Effects." *Journal of Financial Markets*, vol. 5, no. 1 (January):31–56.

Ané, T., and H. Geman. 2000. "Order Flow, Transaction Clock, and Normality of Asset Returns." *Journal of Finance*, vol. 55, no. 5 (October):2259–2284.

Bangia, A., F.X. Diebold, T. Schuermann, and J. Stroughair. 2002. "Modeling Liquidity Risk, with Implications for Traditional Market Risk Measurement and Management." In *Risk Management: The State of the Art*. Edited by S. Figlewski and R. Levich. Boston: Kluwer Academic.

Bertsimas, D., and A. Lo. 1998. "Optimal Control of Execution Costs." *Journal of Financial Markets*, vol. 1, no. 1 (April):1–50.

Brigo, D., and C. Nordio. 2010. "Liquidity-Adjusted Market Risk Measures with Stochastic Holding Period." Working paper (October).

Çetin, U., R. Jarrow, and P. Protter. 2004. "Liquidity Risk and Arbitrage Pricing Theory." *Finance and Stochastics*, vol. 8, no. 3 (August):311–341.

Engle, R., and R. Ferstenberg. 2007. "Execution Risk." *Journal of Portfolio Management*, vol. 33, no. 2 (Winter):34–44.

Gatheral, J. 2010. "No-Dynamic-Arbitrage and Market Impact." *Quantitative Finance*, vol. 10, no. 7 (August):749–759.

Gatheral, J., A. Schied, and A. Slynko. 2012. "Transient Linear Price Impact and Fredholm Integral Equations." *Mathematical Finance*, vol. 22, no. 3 (July):445–474.

Gourieroux, C., J.P. Laurent, and O. Scaillet. 2000. "Sensitivity Analysis of Values at Risk." *Journal of Empirical Finance*, vol. 7, no. 3–4 (November):225–245.

Grinold, R.C., and R. Kahn. 1999. *Active Portfolio Management: A Quantitative Approach for Producing Superior Returns and Controlling Risk*. 2nd ed. New York: McGraw-Hill.

Hallerbach, W. 2003. "Decomposing Portfolio Value-at-Risk: A General Analysis." *Journal of Risk*, vol. 5, no. 2 (Winter):1–18.

Hibbert, J., A. Kirchner, G. Kretzschmar, R. Li, A. McNeil, and J. Stark. 2009. "Summary of Liquidity Premium Estimation Methods." Research report, Barrie & Hibbert (October).

Jorion, P. 2007. *Value at Risk*. New York: McGraw-Hill.

Lo, A., C. Petrov, and M. Wierzbicki. 2003. "It's 11pm—Do You Know Where Your Liquidity Is? The Mean-Variance Liquidity Frontier." *Journal of Investment Management*, vol. 1, no. 1 (First Quarter):55–93.

Lobo, M., M. Fazel, and B. Boyd. 2007. "Portfolio Optimization with Linear and Fixed Transaction Costs." *Annals of Operations Research*, vol. 152, no. 1 (July):341–365.

Meucci, A. 2005. *Risk and Asset Allocation*. Berlin: Springer.

———. 2010a. "Personalized Risk Management: Historical Scenarios with Fully Flexible Probabilities." *Risk Professional* (December):47–51.

———. 2010b. "Return Calculations for Leveraged Securities and Portfolios." *Risk Professional* (October):40–43.

———. 2011a. "Mixing Probabilities, Priors, and Kernels via Entropy Pooling." *Risk Professional* (December):32–36.

———. 2011b. "The Prayer: The 10 Steps of Advanced Risk and Portfolio Management." *Risk Professional* (April):54–60.

———. 2011c. "The Prayer: The 10 Steps of Advanced Risk and Portfolio Management—Part 2." *Risk Professional* (June):34–41.

Obizhaeva, A., and J. Wang. 2005. "Optimal Trading Strategy and Supply/Demand Dynamics." NBER Working Paper w11444 (June).

Tasche, D. 2002. "Expected Shortfall and Beyond." *Journal of Banking & Finance*, vol. 26, no. 7 (July):1519–1533.

Toth, B., Y. Lemperiere, C. Deremble, J. De Lataillade, J. Kockelkoren, and J.P. Bouchaud. 2011. "Anomalous Price Impact and the Critical Nature of Liquidity in Financial Markets." Working paper (May).