Minimum-Variance Portfolio Composition

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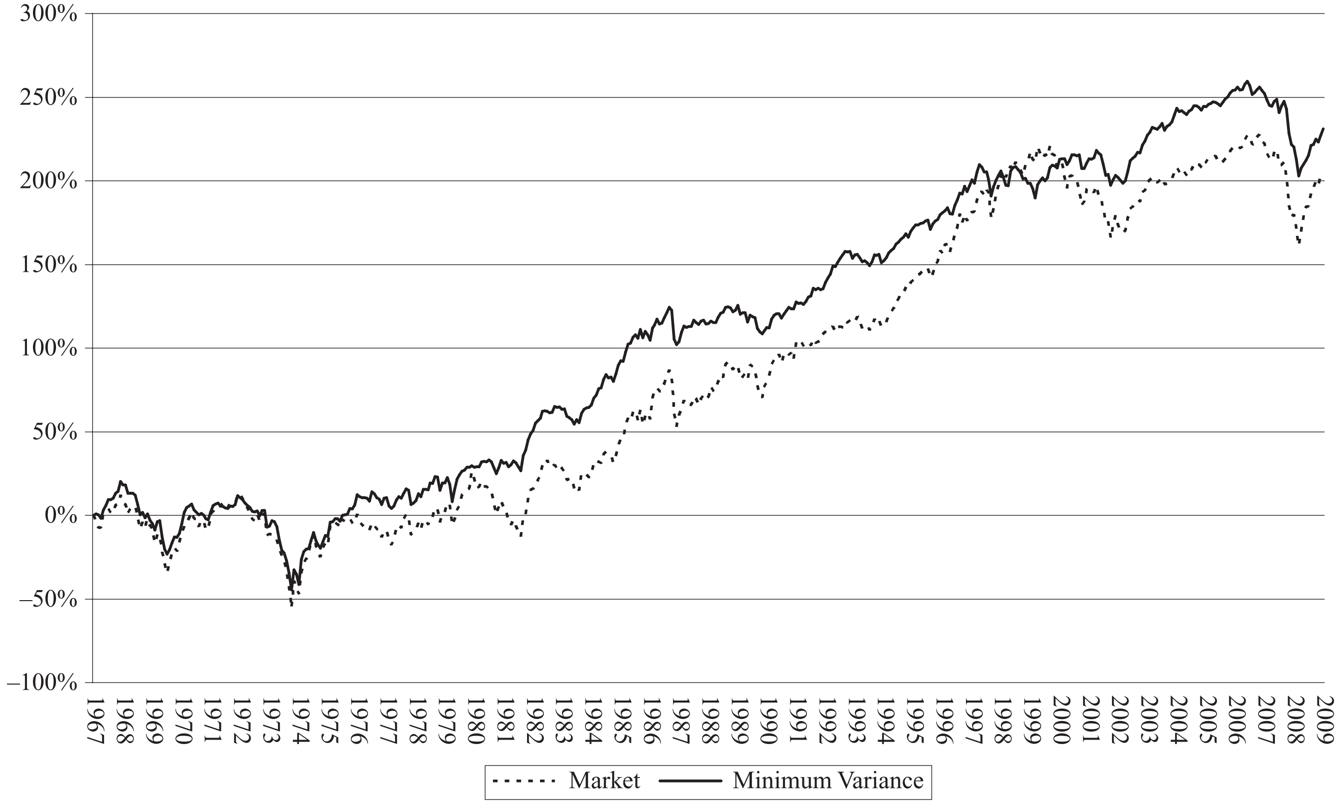
The performance of equity portfolios optimized to have the lowest possible variance has attracted investor attention over the last several years. Minimum-variance strategies address an increased appreciation for risk management due to the financial crisis, as well as to the historical fact that low-volatility stocks tend to have returns that meet or exceed the market. The empirical observation that high-market-beta stocks are not rewarded with correspondingly higher returns is a long-standing empirical critique of the CAPM, as summarized by Fama and French [1992]. More recently, Ang et al. [2006] documented a low-risk, high-return empirical anomaly that they associate with idiosyncratic risk. The popularity of low-volatility portfolio strategies that exploit these and other potential equity market anomalies prompted MSCI to launch the Global Minimum Volatility Indices for various geographic regions in August 2008.

The relative return to low-volatility portfolios has continued to be strong in recent years. Exhibit 1 is a chart of the cumulative excess returns from 1968 to 2009 on two portfolios composed of large-cap U.S. equities, similar to the portfolios studied by Clarke, de Silva, and Thorley [2006].[1] The dotted line in the exhibit is the cumulative return to the market, measured by the capitalization-weighted portfolio of the largest 1,000 stocks. The solid line in the exhibit is the cumulative return to a long-only minimum-variance portfolio constructed from the same set of securities. As shown on the right side of Exhibit 1, the cumulative excess return of the minimum-variance portfolio has been slightly higher than cumulative excess return of the market over the past 42 years. Despite the higher average return, the realized risk of the minimum-variance portfolio is well below that of the market as indicated by less fluctuation in the solid line compared to the dotted line. For example, the impact of the October 1987 market crash is somewhat muted, and the large decline in the general market after the turn of the century appears as only a minor decline in the minimum-variance portfolio. The successful creation of a low-realized-risk portfolio simply confirms the value of security risk forecasting and the process of portfolio optimization. But if the relatively high return to low-risk portfolios continues into the future, the result represents a valuable investment opportunity in addition to a puzzling violation of risk–return principles in financial economics.

In this article, we examine the composition of minimum-variance portfolios with a focus on the analytic form and parameter values of individual security weights. Efficient frontiers in mean-variance optimization are typically described using the mathematics of *unconstrained* portfolios, but implemented in a long-only *constrained* format using a numerical optimizer. Although the

Exhibit 1

Market and Minimum-Variance Portfolios' Cumulative Returns of 1,000 Largest U.S. Stocks, 1968–2009



mathematics of unconstrained optimization is well known, a key innovation in this article is an analytic solution for optimal security weights under the long-only constraint. Constrained solutions in portfolio mathematics are generally non-tractable, but the assumption of a single-factor risk model allows for a simple and intuitive equation for the optimal weights. The portfolio mathematics shows that while high estimated idiosyncratic risk can lead to a low security weight, high systematic risk takes the large majority of investable securities completely out of the long-only solution.

The analytic results from the single-factor model indicate that minimum-variance portfolios are strictly populated by stocks with betas lower than a specified threshold. Security weight as well as membership in the optimized long-only portfolio is less sensitive to idiosyncratic risk, an intuitive result from portfolio theory. We compare the single-factor analytics with numerical optimizations on a more general covariance matrix and find that the consideration of non-market sources of security correlation only marginally modifies the analytically derived optimal weights. Further development of the optimization mathematics shows that the ratio of portfolio beta to the long-only threshold beta dictates the portion of ex ante portfolio variance related to market exposure. Values of this ratio over time indicate that 80% to 90% of long-only minimum-variance portfolio risk is systematic in the single-factor model. Together, the analytic and empirical findings suggest that the strong performance of minimum-variance portfolios is related to the long-standing empirical critique of the CAPM that low-beta stocks have relatively high returns. To the extent that it measures a separate phenomenon, the more recently identified idiosyncratic risk anomaly of domestic equity markets by Ang et al. [2006] and in international equity markets by Ang et al. [2009] is less likely to impact portfolio returns.

The first section of this article reviews the performance statistics of the market and minimum-variance portfolios reported in Exhibit 1, as well as two variations of the base-case minimum-variance portfolio. The second section discusses the analytic results derived in the technical appendix with a focus on the relatively simple equation that emerges for long-only optimal security weights. The third section illustrates the application of the analytic results using empirical data on 1,000 U.S. stocks from 1968 to 2009, followed by a detailed analysis of the optimization for January 2010. The fourth section examines the decomposition of ex ante portfolio risk into benchmark exposure and residual risk over time. The final section summarizes conclusions about minimum-variance portfolio composition and discusses implications for general mean-variance optimization in a long-only setting.

PORTFOLIO STATISTICS AND COVARIANCE MATRIX ESTIMATION

In this section we review the portfolio statistics for the market and minimum-variance portfolios shown in Exhibit 1 and discuss our security covariance matrix estimation methodology. The first two columns of Exhibit 2 show return statistics for the market and what we hereafter refer to as the base-case minimum-variance portfolio. The statistics in Panel A of Exhibit 2 are calculated from monthly observations and reported as annualized (multiplied by 12) returns in excess of the T-bill return, which averaged about 5% over the period 1968–2009. For expositional simplicity we hereafter drop the adjectives "excess" and "annualized," although they apply to all returns reported in this study. The 5.37% average return of the base-case minimum-variance portfolio is slightly higher than the market benchmark return of 4.88%. The realized risk as measured by a standard deviation of 11.90% is only about three-fourths of the market portfolio's risk of 15.56%. The result in Panel A is a Sharpe ratio of 0.45 for the base-case minimum-variance portfolio compared to only 0.31 for the market portfolio. Specifically, matching the base-case portfolio risk of 11.90% with a risk-equivalent portfolio of a 11.96/15.56 = 77% market index and a 23% risk-free cash investment yields a 5.37 – (0.77 × 4.88) = 161 basis-point advantage to the minimum-variance portfolio.

Panel B of Exhibit 2 shows that the average market beta of the base-case portfolio is 0.66, calculated by a single regression of the minimum-variance returns on market returns over the entire 504-month history. The annualized intercept term from this regression, which we refer to as alpha, is 2.17% for the base-case minimum-variance portfolio.[2] The active risk (annualized standard deviation of residual return) is 6.10%, indicating that the reduction in total portfolio risk comes with substantial residual risk to the benchmark. The information ratio (alpha divided by active risk) of the base-case portfolio is 0.35 compared to zero, by definition, for the market index. Although benchmark relative risk and return is not the objective of minimum-variance optimization, information ratios of this magnitude over four decades are rare, even for strategies that target risk-adjusted active return.

For comparison purposes, the third column in Exhibit 2 reports the returns for an unconstrained long–short minimum-variance portfolio constructed from the same universe of stocks as the long-only base case. The realized risk of 10.61% for the long–short portfolio is lower than that of the base case due to relaxing the long-only constraint, but the realized average return is also lower leading to a similar Sharpe ratio. The long–short optimization assigns positive and negative weights to all 1,000 securities each month, with an average of about 175% of the notional value long and 75% short, or what could be referred to as a 175/75 portfolio. Shorting of this magnitude is costly or impractical in many applied settings, so the long-only base-case portfolio remains the focus of this study. The base-case portfolio averages about 120 long security positions over time (i.e., about 12% of

EXHIBIT 2
Annualized Portfolio Excess Returns and Performance, 1968–2009

		Minimum Variance		
	Market	Long-only Base Case	Long–Short	Long-only Single Index
Panel A: Statistics				
Average	4.88%	5.37%	4.45%	5.50%
Standard Deviation	15.56%	11.90%	10.61%	12.83%
Sharpe Ratio	0.31	0.45	0.42	0.43
Panel B: Regression				
Beta	1.00	0.66	0.51	0.55
Alpha	0.00%	2.17%	1.97%	2.82%
Active Risk	0.00%	6.10%	7.07%	9.58%
Information Ratio	0.00	0.35	0.28	0.29

the 1,000-security investable set) and has a security weight distribution similar to the market portfolio. For example, the maximum security weight in the January 2010 optimization, which will be discussed in detail later, is about 3%, and the large majority (86 of the 128 positions for that month) is below 1%. The return statistics for the long-only single-index minimum-variance portfolio reported in the last column of Exhibit 2 is discussed in the next section.

Although we refer to *the* minimum-variance portfolio, uniqueness is specific to the ex ante, or predicted, security covariance matrix supplied to the optimization routine, and estimation techniques vary. The estimation of the ex ante security covariance matrix used in this study employs an intentionally simple process: a 60-month rolling window of individual security return variance and covariance terms.[3] The 1,000 variance terms along the diagonal of the 1,000-by-1,000 matrix as well as the 499,500 unique off-diagonal covariance terms are "shrunken" towards their respective cross-sectional means each month in accordance with the Bayesian statistical theory of Ledoit and Wolf [2004].[4] Although more sophisticated covariance matrix estimation techniques (e.g., factor models and GARCH) are available, we use the raw historical sample data to keep our empirical conclusions generic and replicable. The monthly optimization process is also generic in that no constraints are imposed except the restrictions that individual security weights are positive (long-only constraint) and sum to 100% (full investment constraint). The Bayesian adjustment of the security covariance matrix elements leads to maximum security weights of 3% to 4% in most months without the need for individual position constraints.[5]

MINIMUM-VARIANCE PORTFOLIO SECURITY WEIGHTS FOR THE SINGLE-INDEX MODEL

Prior empirical studies document most of the long-term minimum-variance portfolio results discussed in the preceding section. Historical backtesting and simulation are subject to data-mining biases, however, and provide only a limited perspective on portfolio composition. In this section, we explore the mathematics of minimum-variance portfolios with the objective of providing a deeper analytic understanding of the long-term results. We employ Sharpe's [1963] well-known single-index assumption that the only source of common risk across equity securities is a single factor—the capitalization-weighted market portfolio. A key analytic result derived in the appendix is that, under the assumption of the single-index model for security returns, the weight for individual securities in the unconstrained minimum-variance portfolio is

$$w_i = \frac{\sigma_{MV}^2}{\sigma_{\varepsilon i}^2}\left(1 - \frac{\beta_i}{\beta_{LS}}\right) \tag{1}$$

where

σ_{MV}^2 = ex ante return variance of the minimum-variance portfolio
$\sigma_{\varepsilon i}^2$ = ex ante idiosyncratic return variance for security i
β_i = ex ante market beta for security i
β_{LS} = long–short threshold beta

The relatively simple form of Equation (1) is attributable to the absence of expected returns in the minimum-variance objective function and the assumption of a single-index risk model. The full specification for the long–short threshold beta, β_{LS}, discussed in the appendix, turns out to be just slightly higher than the average beta across all investable securities. The position of this long–short threshold beta in the denominator of the second term of Equation (1) indicates that securities with betas greater than β_{LS} (a little less than half of the investable securities) are assigned negative weights in unconstrained optimizations. A key insight from Equation (1) is that systematic rather than idiosyncratic risk dictates whether an individual security has a negative weight in an unconstrained optimization.

An even more novel mathematical result from the appendix is that the basic form of Equation (1) is preserved in long-only constrained optimizations,

$$w_i = \frac{\sigma_{LMV}^2}{\sigma_{\varepsilon i}^2}\left(1 - \frac{\beta_i}{\beta_L}\right) \text{ for } \beta_i < \beta_L \text{ else } w_i = 0 \tag{2}$$

where

σ_{LMV}^2 = ex ante return variance of the long-only minimum-variance portfolio
β_L = long-only threshold beta

Equation (2) holds for long-only constrained portfolios that are generally not subject to closed-form mathematical analysis. As in Equation (1), individual security weights depend on two portfolio-wide parameters: the ex ante variance, σ^2_{LMV}, and the long-only threshold beta, β_L. The cross-sectional variation in the individual positive weights is driven by the two security-specific risk parameters: market-model idiosyncratic variance, $\sigma^2_{\varepsilon i}$, and market beta, β_i. High idiosyncratic volatility in the denominator of the first term of Equation (2) can drive the security weight towards zero, but not out of solution. High market beta in the second term also leads to a lower security weight, but when the market beta is larger than the long-only threshold beta the security is assigned an optimal weight of zero. The long-only threshold beta typically falls within the lowest-beta quintile of the investable set, so that a large majority of securities, for example 80%, have zero weights. The 20% of all investable securities in the long-only optimal solution are those with the lowest estimated betas.

Equation (2) holds exactly under a simplified covariance matrix structure—a risk model in which the only source of correlation is through a single common factor. For purposes of comparison, the final column of Exhibit 2 reports the track record of a long-only minimum-variance portfolio that is actually optimized from a single-index model covariance matrix. We formed our single-factor covariance matrix directly from the Bayesian-adjusted sample data, but also populated the matrix using OLS regressions of individual security returns on the market with little change in the results.[6] The single-index portfolio realized risk of 12.83% in Exhibit 2 is slightly higher than that of the base-case portfolio, presumably due to sources of security-to-security correlation not captured by the single factor. The single-index minimum-variance portfolio results verify Equation (2) in that the output of the numerical optimizer exactly matches the security weights calculated analytically.

CROSS-SECTIONAL EXAMINATION OF SECURITY RISK AND WEIGHTS

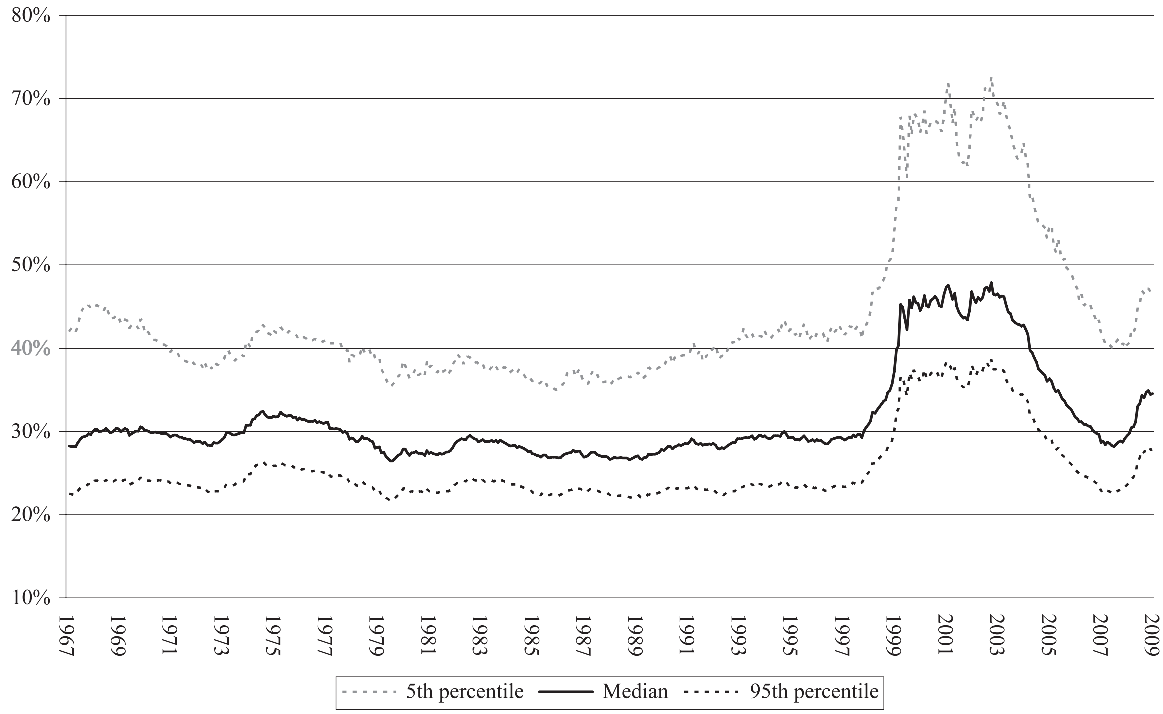
According to Equations (1) and (2), the two security-specific parameters that determine the cross-sectional variation in optimal security weights under the single-index model are beta and idiosyncratic risk. In this section, we document the historical ranges of these security risk parameters and illustrate their impact on security weights. Exhibits 3 and 4 plot the 5th–95th percentile range and

EXHIBIT 3

Range of Forecasted Market Betas of 1,000 Largest U.S. Stocks, 1968–2009

Exhibit 4

Range of Forecasted Idiosyncratic Risks of 1,000 Largest U.S. Stocks, 1968–2009



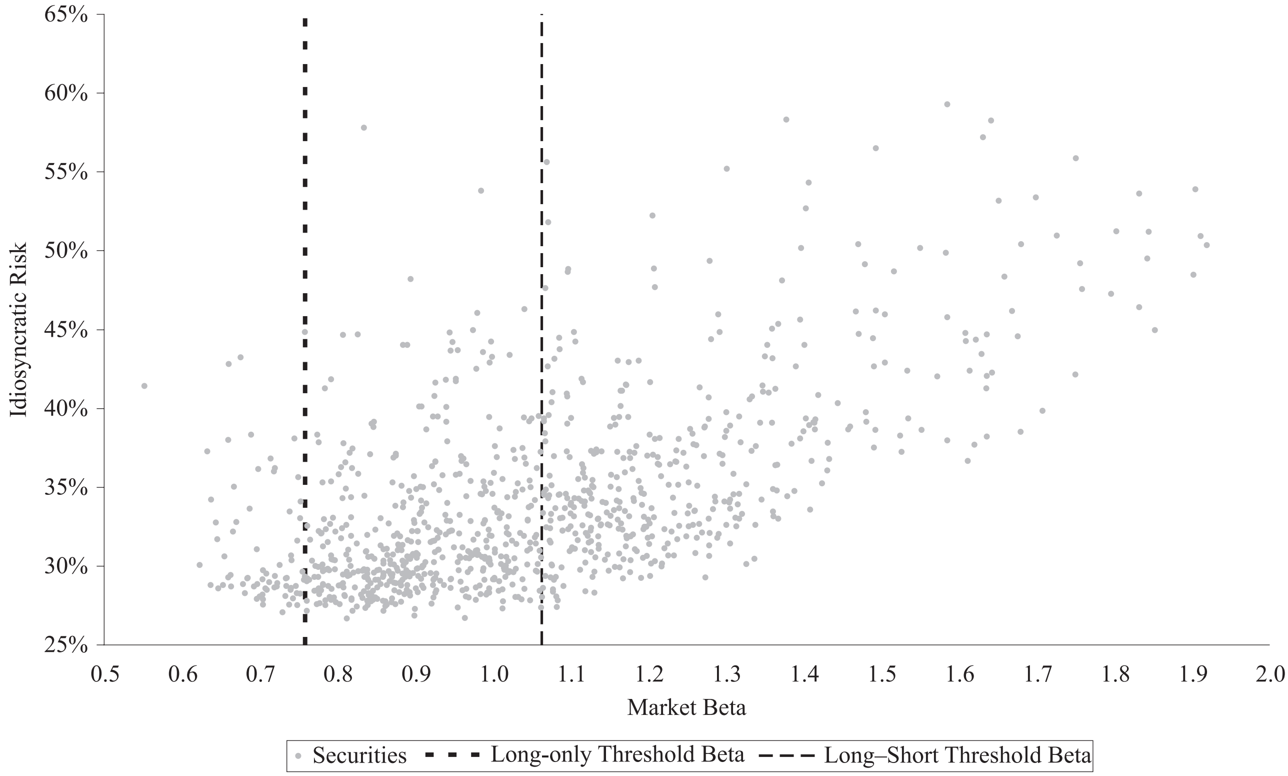
median value for 1,000 large-cap U.S. stocks from January 1968 to December 2009. The median security beta in Exhibit 3 varies around one, but is not exactly 1.0 as it would be in a capitalization-weighted average. The 5th–95th percentile range of betas can be characterized as about 0.8–1.4 over time, except for around the turn of the century when the lower end dipped to about 0.5 and then a few years later the upper range increased to about 1.7. Note that changes in the range of beta reported in Exhibit 3 are smoothed by the use of a rolling 60-month window of historical returns data; a shorter window would be more reactive to market events. Also note that Exhibit 3 plots the range of *forecasted* betas; the range of realized betas across the 1,000 stocks is about twice as large. The Bayesian adjustment of historical betas by one-half towards their mean for purposes of forecasting is motivated by the previously mentioned Bayesian statistical theory and is consistent with cross-sectional regressions of realized beta on historical betas.[7]

Exhibit 4 shows the cross-sectional median and 5th–95th percentile range for the 1,000 security ex ante idiosyncratic standard deviations at the beginning of each month. Median idiosyncratic risk fluctuates around 30%, except for several years after the turn of the century when the median increased to about 45%. The cross-sectional spread of idiosyncratic security risk is fairly stable over time, with a 5th–95th percentile range of about 25% to 40%, except for a sharp rise after the turn of the century. As with the market betas, Exhibit 4 plots the range of forecasted rather than realized 60-month residual risks, which have a spread about 1.4 (square root of 2) times larger.

Given the simplicity of the analytic solution in Equations (1) and (2), one way to study the impact of cross-sectional variations in security risk is to plot the security weights in a minimum-variance portfolio for a specific example month. Exhibit 5 is a scatterplot of the 1,000 forecasted values for the two market-model security risk parameters for January 2010. With the exception of a few high values not included in Exhibit 5, the security betas fall between 0.5 and 2.0. Exhibit 5 shows vertical lines for the long-only threshold beta of 0.76 and the long–short threshold beta of 1.06 for this particular month, calculated according to Equations (A5) and (A6) in the appendix. The range for forecasted idiosyncratic security risk on the vertical axis of Exhibit 5 is about 25% to 65%, consistent with the right side of Exhibit 4. Exhibit 5

EXHIBIT 5

Single-Index Model Risk Parameters for 1,000 U.S. Stocks, January 2010



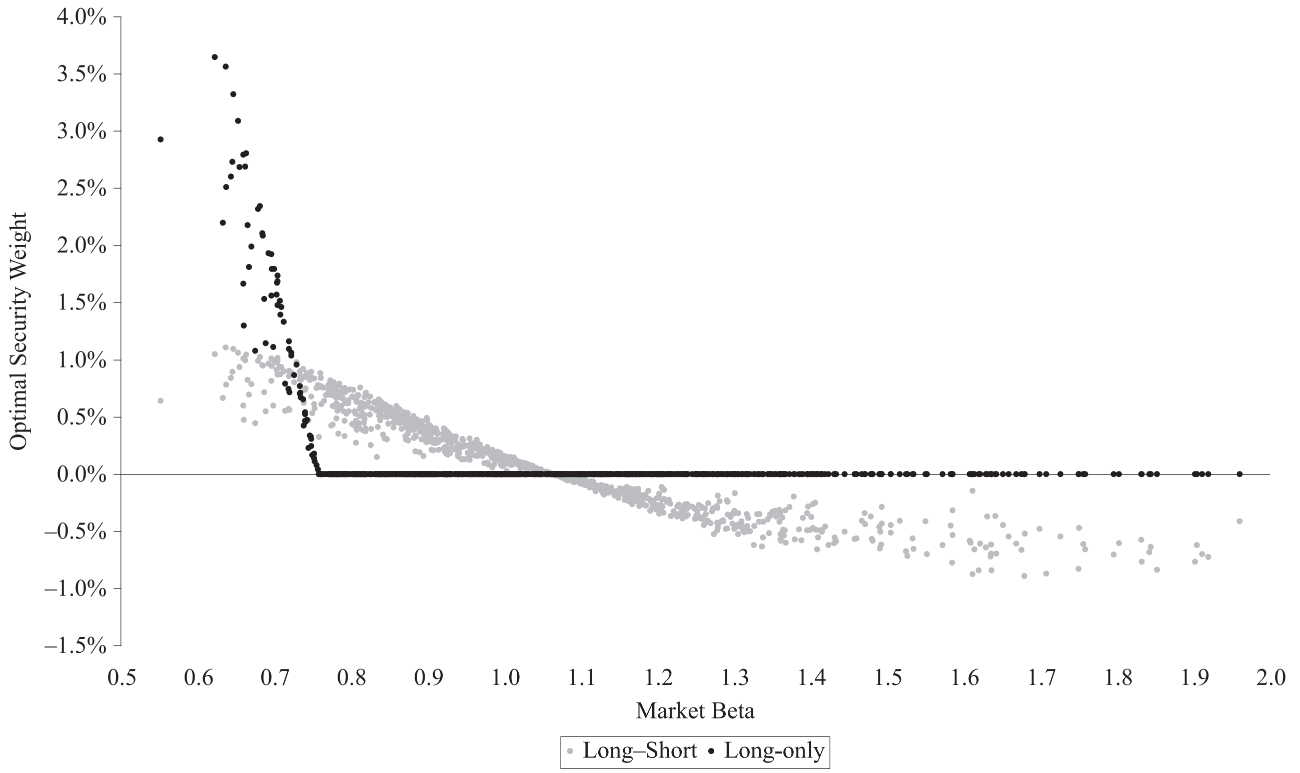
verifies the well-known positive correlation between idiosyncratic risk and market beta, at least for mid-range- and higher-beta stocks. A less appreciated empirical property is a slightly *negative* correlation between idiosyncratic risk and market beta for the low-beta stocks that are included in the long-only minimum-variance portfolio (i.e., to the left of the long-only threshold).

Exhibit 6 shows the 1,000 individual security weights under the single-index model for both the long–short (grey dots) and long-only (black dots) minimum-variance portfolios plotted against market beta for January 2010. The optimal long–short weights (grey dots) decline with higher market beta in accordance with the role of β_i in Equation (1), and move from positive to negative values as the security beta exceeds the long–short threshold value of 1.06. The optimal long-only weights (black dots) also decline with market beta, but stop at zero for beta values above the long-only threshold beta of 0.76, in accordance with Equation (2). In fact, only 71 of the 1,000 investable securities are in the long-only solution for January 2010. For these securities, the decline in long-only weights is steeper than the decline in long–short weights because the long-only threshold beta is lower than the long–short threshold beta. The long–short weights (grey dots) in Exhibit 6 deviate from a strict line due to the idiosyncratic risk parameter in Equation (1). The deviations are towards lower values for large-positive-weight securities and higher values for large-negative-weight securities. The long-only weights (black dots) also tend to deviate from a strict kinked line with lower optimal weights due to the slightly negative correlation between beta and idiosyncratic risk for the stocks that come into solution.

Exhibit 7 shows the 1,000 optimal security weights under the single-index model for the long–short (grey dots) and long-only (black dots) portfolios plotted against idiosyncratic risk. While there is some tendency for lower-idiosyncratic-risk securities to have positive long–short weights (grey dots), the clearest pattern is that high-idiosyncratic-risk securities have low-absolute-value weights, consistent with the role of idiosyncratic risk in Equation (1). A pattern of lower long-only weights (black dots) for higher-idiosyncratic-risk securities is only slightly

EXHIBIT 6

Minimum-Variance Security Weights and Market Beta Single-Index Optimization, January 2010



evident in Exhibit 7. Only low-beta securities are present in the long-only portfolio, and as shown in Exhibit 5, idiosyncratic risk is not positively correlated to beta for these securities. The key contrast between Exhibits 6 and 7 is that optimal security weights are much more aligned with security beta than idiosyncratic risk, at least when the optimization is based on the single-index covariance matrix assumption that underlies Equations (1) and (2).

Exhibit 8 is similar to Exhibit 6, but plots security weights for the general covariance matrix optimizations reported in the second and third columns of Exhibit 2. The general rather than single-index covariance matrix used in the construction of the base-case minimum-variance portfolio incorporates correlation patterns beyond the single market factor that can drive security weights away from the analytic values specified in Equation (2). The weights still decline with market beta in Exhibit 8, but the pattern is not as tight as in Exhibit 6 where the only source of covariance between securities is through the market factor. But the key principle that low-market-beta stocks dominate the long-only minimum-variance portfolio is still quite evident in Exhibit 8. All of the 128 (compared to 71 in Exhibit 6) long-only weights are for securities with betas less than one, and most of the larger-weight securities have betas below the long-only threshold value of 0.76. The companion plot (not shown) to Exhibit 8 for weights and idiosyncratic risk using the general covariance matrix is not materially different from Exhibit 7.

The concentration of low-beta stocks in long-only optimizations is consistent with recently published indices by MSCI for minimum-volatility portfolios. For example, at the end of 2009, 61% of the MSCI World Minimum Volatility Index was in the lowest-beta-quintile stocks, an additional 24% was in the stocks of the second-lowest beta quintile, for a total of 85%. By contrast, the standard capitalization-weighted MSCI World Index is more evenly distributed across the beta quintiles. Specifically, at the end of 2009, 25% of the MSCI World Index was in the lowest-beta-quintile stocks, and another 22% was in the stocks of the second-lowest beta quintile, for a total of 47%.[8]

Exhibit 7

Minimum-Variance Security Weights and Idiosyncractic Risk Single-Index Optimization, January 2010

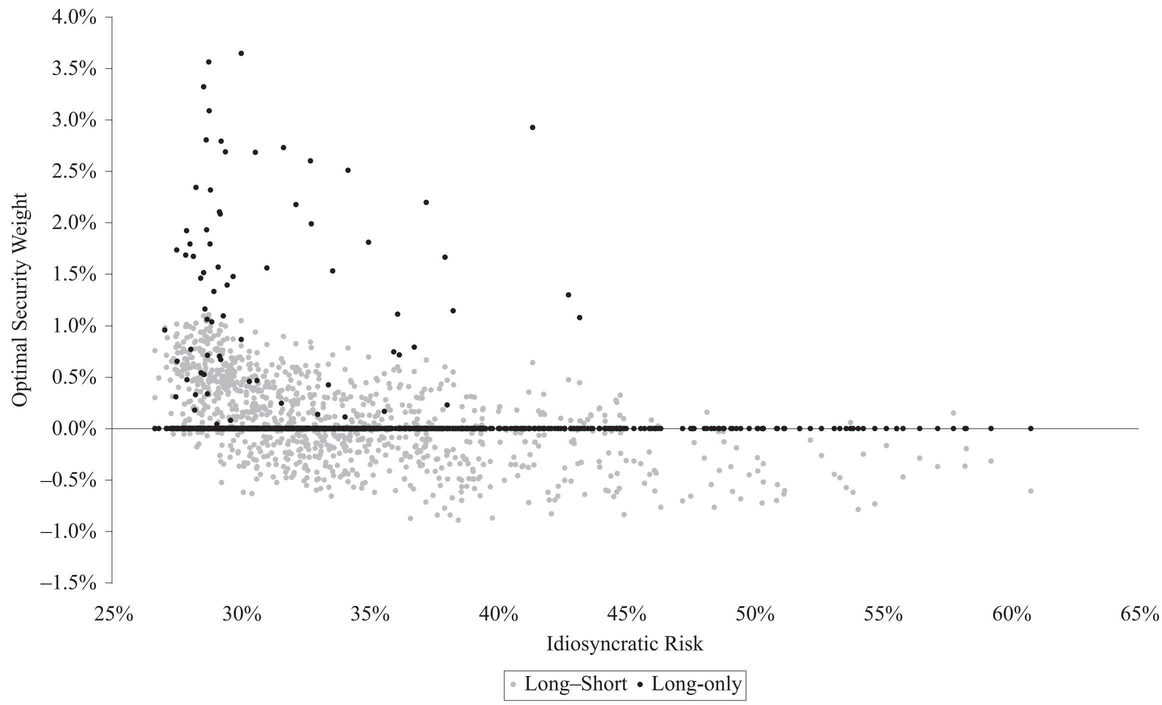
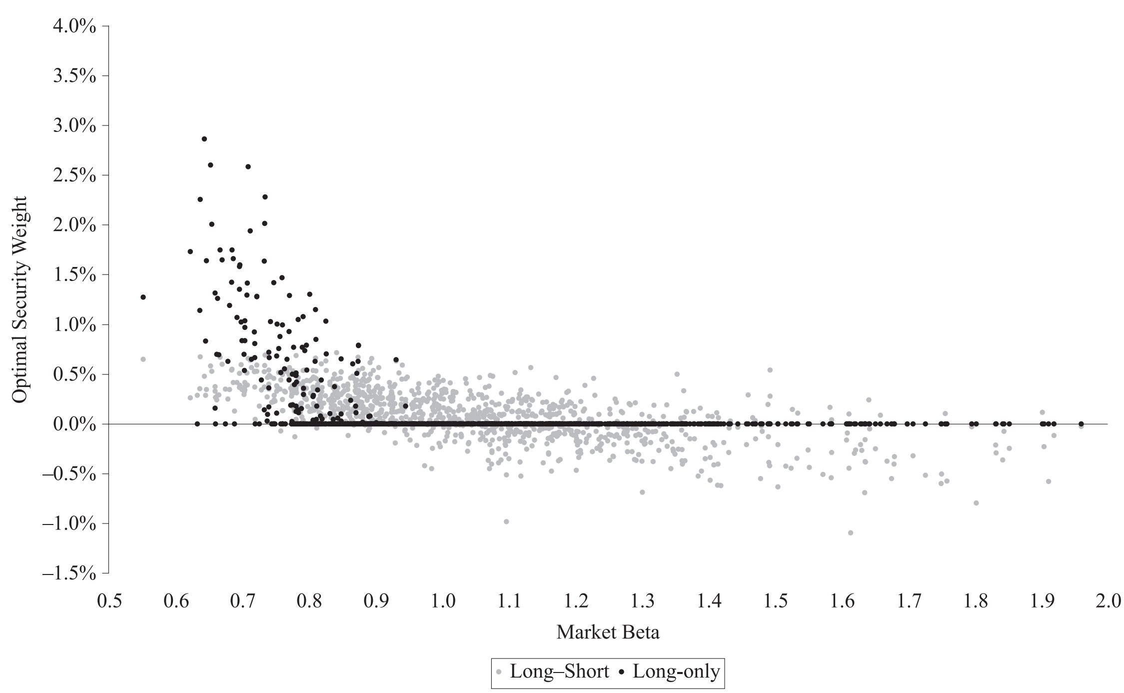


Exhibit 8

Minimum-Variance Security Weights and Market Beta Base-Case Optimization, January 2010



TEMPORAL DYNAMICS AND DECOMPOSITION OF EX ANTE PORTFOLIO RISK

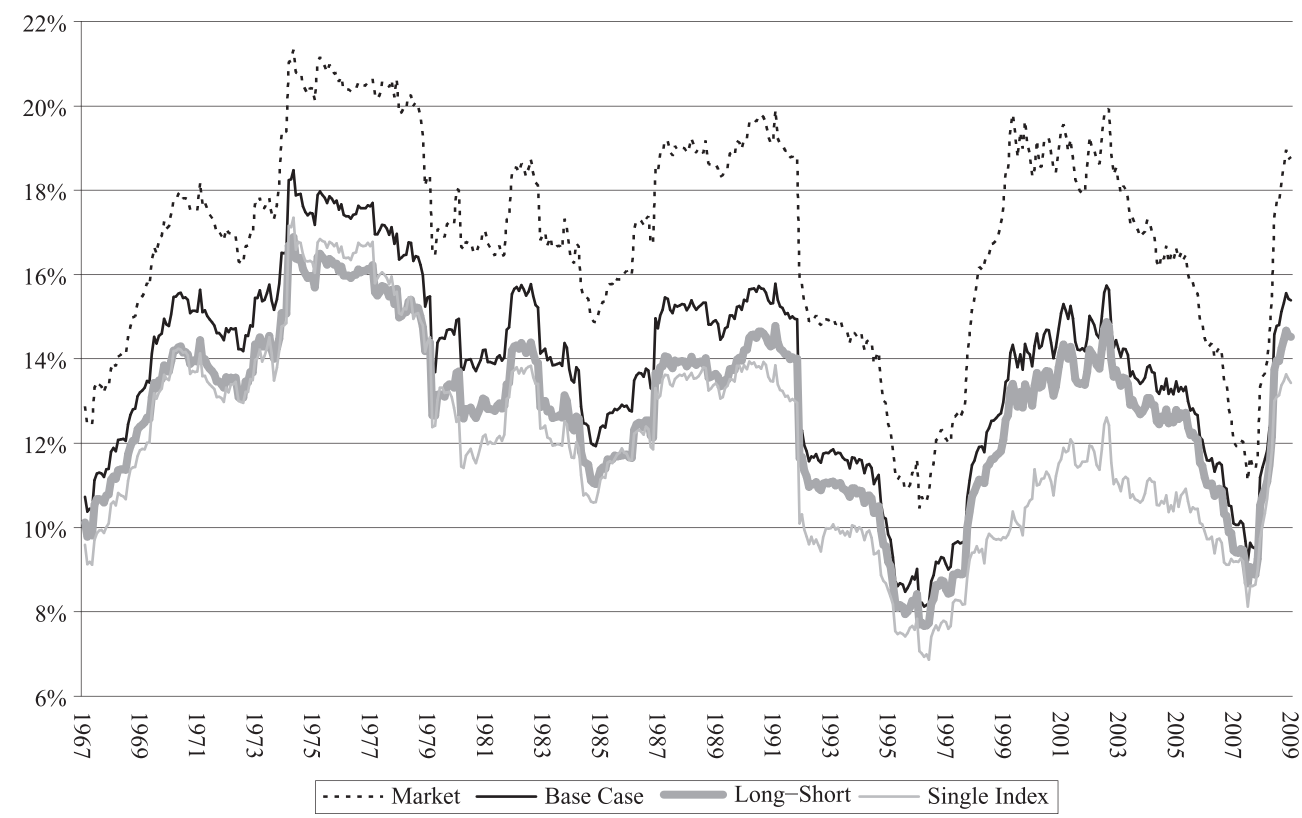
The portfolio statistics and time series regressions over the entire 42-year sample in Exhibit 2 mask substantial intertemporal dynamics in minimum-variance portfolio risk. In this section, we examine the ex ante risks for each optimization over time to see how minimum-variance portfolios adapt to changing market conditions. Exhibits 9 and 10 are based on the ex ante portfolio return standard deviation estimated at the beginning of each of the 504 months for the four portfolios in Exhibit 2. By design, the ex ante risk forecasts are similar to the realized risk of each portfolio over the prior 60 months. For example, the sudden drop in forecasted risk for all four portfolios towards the end of 1992, as shown in Exhibit 10, is a result of the large negative October 1987 return observations dropping out of the 60-month rolling window. The highest ex ante risk is from the non-optimized market portfolio in Exhibit 10 (dotted line), with an average value of 16.88% as reported in Exhibit 9. The ex ante risk for the base-case minimum variance portfolio in Exhibit 10 (solid line) moves with the market portfolio, but is, by construction, always below the market.

Consistent with the ordering of realized risks in Exhibit 2, Exhibit 10 shows that the long–short minimum-variance portfolio's ex ante risk (double line) is always slightly below the risk of the base case because of relaxing the long-only constraint. In contrast, the

EXHIBIT 9
Portfolio Ex Ante Risk, 1968–2009

	Market	Long-only Base Case	Long–Short	Long-only Single Index
Average	16.88%	13.78%	12.78%	12.01%
Min	10.46%	8.12%	7.67%	6.87%
Max	21.33%	18.47%	16.89%	17.35%

EXHIBIT 10
Ex Ante Portfolio Risk (Standard Deviation), 1968–2009



long-only single-index portfolio's ex ante risk (grey line) is lower than the risk of the other portfolios shown in Exhibit 9 in contrast to the higher *realized* risk shown in Exhibit 2. The single-index model's ex ante risk tends to understate the actual realized risk of that portfolio because the covariance matrix does not account for correlation structures beyond the single risk factor. The average ex ante risk of the single-index portfolio measured by the general covariance matrix (used in the construction of the base-case portfolio) is 14.66%, which is both more realistic (i.e., closer to) and consistent with the relative ordering of the realized portfolio risks given in Exhibit 2.

A common practice in portfolio management is to attribute total portfolio risk to various factor exposures and idiosyncratic risk. As discussed in the appendix, the process of linear factor risk decomposition is based on variances rather than standard deviation and is exhaustive in that it apportions all of the risk to one or another source. The appendix derives the interesting result that the systematic portion of long-only minimum-variance portfolio risk under the single-index model is equal to the ratio of the ex ante portfolio beta and the long-only threshold beta,

$$\frac{\beta_P^2 \sigma_M^2}{\sigma_P^2} = \frac{\beta_P}{\beta_L} \tag{3}$$

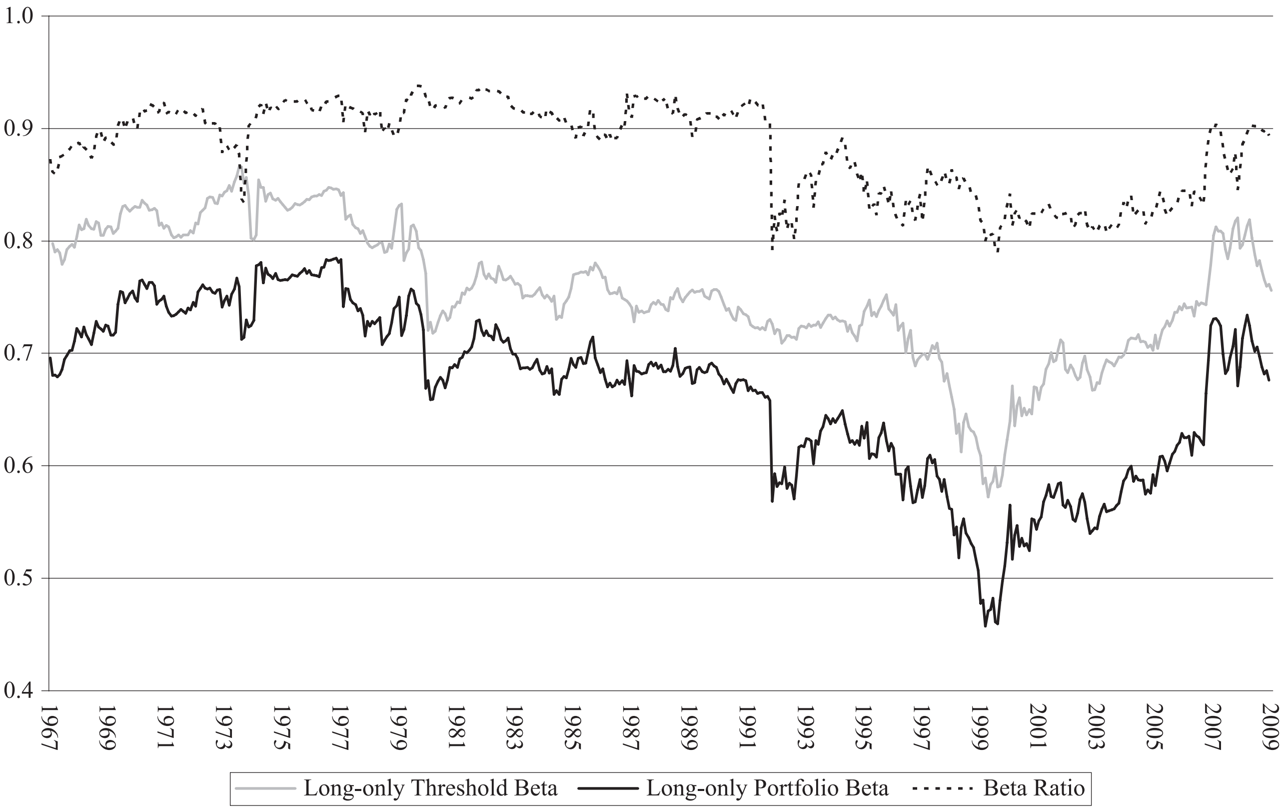
For example, if the long-only threshold beta is $\beta_L = 0.8$, then the long-only minimum-variance portfolio is composed of securities with betas below 0.8. If the resulting portfolio beta (weighted average of security betas) is $\beta_P = 0.7$, then Equation (3) states that seven-eighths of the ex ante portfolio variance is associated with market exposure, leaving only one-eighth of the total portfolio risk as idiosyncratic.

Exhibit 11 plots the long-only threshold and single-index minimum-variance portfolio betas over time, showing values that swing around 0.8 and 0.7, respectively, at the beginning of 1968, to lower than 0.6 and 0.5, respectively, at the turn of the century, and then back again. The ratio of beta values specified by Equation (3), shown as the dotted line at the top of Exhibit 11, indicates that the portion of ex ante portfolio risk attributable to market exposure varies over time and has an average value of about 90%. For example, when the October 1987 observation drops out of the 60-month risk model in November 1992, the portion of optimized portfolio risk designated as systematic suddenly drops from over 90% to about 80%, and remains closer to 80% until observations from the 2008 financial crisis begin to be included in the risk model. The use of ex ante risk forecasts reveals the dynamic and changing nature of equity market risk over time, but one consistent result is that a large majority (80%–90%) of the analytically tractable long-only minimum-variance portfolio risk is associated with general market exposure.

Because of the principle of diversification, the minimum-variance portfolio risk decomposition discussed earlier is essentially the opposite of the risk decomposition of the individual securities that compose the portfolio. For example, Exhibits 3 and 4 indicate that a typical U.S. stock has a beta of about 1.0 and idiosyncratic risk of about 30%. If the market risk is, say, 15%, then the systematic variance of the typical stock is $0.15^2 = 0.0225$, and the idiosyncratic variance is $0.30^2 = 0.0900$, so about 80% of total security risk is idiosyncratic. In addition, the stocks included in the long-only minimum-variance portfolio have an even lower average beta of about 0.66, so their systematic risk is only $(0.66 \times 0.15)^2 = 0.0100$, while their idiosyncratic variance is still about 0.0900. Thus, the variance decomposition of individual stocks in the long-only portfolio is about 10% (0.0100/0.1000) systematic and 90% (0.0900/0.1000) idiosyncratic, just the opposite of the minimum-variance portfolio risk decomposition shown at the top of Exhibit 11. The diversification benefits of a well-constructed portfolio will mostly shift the source of portfolio variance to a systematic market influence even though the individual security variance is dominated by idiosyncratic risk.

Besides an appreciation for the dynamic nature of equity market risks and thus minimum-variance portfolio composition, the key finding of this section is that a large majority of minimum-variance portfolio risk is systematic under the single-index model assumption. While this result is consistent with the basic concept of diversification, the degree to which a long-only optimized portfolio depends on systematic risk is established analytically by Equation (3). Additionally, the realized risk results reported in Exhibit 2 indicate that some of the remaining idiosyncratic risk in the single-index model may be eliminated by numerically optimized portfolios (compare the second and fourth columns) that capture additional correlations in a generalized covariance matrix. Together, these results suggest that the high average return performance of minimum-variance portfolios is best characterized as an exploitation of the classic CAPM critique that the cross-sectional

EXHIBIT 11
Single-Index Minimum-Variance Portfolio



variation in a stock's forecasted beta has no relation to realized average return. The more recently popularized idiosyncratic risk anomaly of Ang et al. [2006] may be indistinguishable from the classic CAPM critique when measured in a portfolio context.[9]

Although the CAPM critique may be a key driver, the dynamic nature of equity risks over time makes it unlikely that simply purchasing a collection of low-market-beta stocks will perform as well as an explicitly optimized portfolio. For example, the realized risk of a capitalization-weighted portfolio of the lowest-beta-quintile stocks (i.e., 200 of 1,000) each month is 12.70%, with an average return of 5.47%. These statistics are similar to the single-index model results reported in Exhibit 2, but less impressive than the base-case optimization. This article employs a rather simple covariance matrix estimation process. More sophisticated proprietary models perform even better in terms of forecasting the covariance structure of security returns, as explained by Chan, Karceski, and Lakonishok [1999].

SUMMARY AND CONCLUSIONS

The minimum-variance portfolio at the left-most tip of the efficient frontier has the unique property that optimal security weights are solely dependent on the security covariance matrix without regard to expected returns. Using the simplification associated with a single-factor model for the security covariance matrix, we were able to derive an analytic solution for the long-only constrained minimum-variance portfolio. The analytic solution together with an empirical examination of security risk parameters for large-capitalization U.S. stocks provides important insights into the composition of minimum-variance portfolios.

According to the single-index model, optimal security weights depend on two individual security risk parameters as well as two portfolio-wide risk parameters. On the one hand, optimal security weights within the portfolio decline with market beta, and only securities with betas below a specific threshold value remain in the long-only solution. On the other hand, high idiosyncratic risk lowers the otherwise optimal security weight, but cannot by itself drive a security out of solution. Plots of security weights for a specific example month indicate that optimal weights are much more aligned with beta-related risk than idiosyncratic risk in accordance with the intuition of optimal diversification. More general, but analytically less tractable, numerical optimizations on a full covariance matrix suggest that systematic risk considerations continue to dominate the construction of minimum-variance portfolios.

The portion of minimum-variance portfolio risk attributable to market exposure is analytically derived under the single-index model and shown to be equal to the ratio of portfolio beta to the long-only threshold beta. While these beta values change over time, the ratio is more stable and indicates that 80% to 90% of long-only minimum-variance portfolio risk is systematic. Numerical generalizations beyond the analytically tractable single-factor model indicate that some of the remaining idiosyncratic risk is eliminated in optimized portfolios. These analytic and empirical results suggest that the surprisingly strong average return performance of minimum-variance portfolios is related to the well-known CAPM critique, which started with Black, Jensen, and Scholes [1972], that low-beta stocks have returns that are indistinguishable from high-beta stocks. Empirical tests of the CAPM using ex ante betas on the large-cap domestic stock market violate a number of assumptions of the equilibrium theory as first noted by Roll [1977]. But the fact that the low-beta, high-return phenomenon was empirically identified in the late 1960s and has not disappeared in the subsequent decades covered by this study suggests robust staying power for the anomaly.

In addition to the specific application to minimum-variance portfolios, the results we have presented in this article have implications for general mean-variance portfolio optimization. First, quantitative portfolio managers often observe that long-only optimized portfolios use only a small subset of the investable securities. Less understood is that the relatively small number of securities in solution does not necessarily come from some complex set of exposure constraints, the interaction of variables in expected return forecasts and the risk model, or from transaction costs and turnover considerations. The optimal security weight equations we have derived in this article show that the variance minimization component in general mean-variance objective functions is sufficient to disqualify a large majority of investable securities.

Second, the mathematics indicates that the many parameter estimates in a large security covariance matrix are not equally important for long-only optimizations. If only the lowest-total-risk quintile (20%) of stocks end up in solution, then the values of only about 4% (20% squared) of all covariance matrix elements directly impact optimal security weights. Thus, a sharper focus on a relatively small subset of risk estimates in the complete covariance matrix may be warranted for long-only mean-variance optimization. Finally, the analytic approach of this study provides further support for employing an explicitly estimated covariance matrix in mean-variance optimization. While the matrix algebra behind analytic derivations in portfolio theory can be daunting, mathematical expressions of optimal weights provide important insights into the otherwise "black box" process of portfolio optimization.

APPENDIX

Optimization Mathematics

In standard Markowitz [1952] portfolio theory, the minimum-variance portfolio has the lowest risk of all possible portfolios, geometrically at the left-most tip of the efficient frontier. The N-by-1 vector of optimal security weights, $\mathbf{w}_{MV}$, only depends on the N-by-N security covariance matrix, $\mathbf{\Omega}$, and not expected security returns. Specifically, the optimization problem is to minimize portfolio variance,

$$\sigma_p^2 = w' \Omega w \tag{A-1}$$

subject to the budget constraint that the sum of the weights is one, $\mathbf{w}'\iota = 1$, where ι is an N-by-1 vector of ones. The matrix calculus solution to this optimization problem is

$$w_{MV} = \frac{\Omega^{-1} \iota}{\iota' \Omega^{-1} \iota} \tag{A-2}$$

where the -1 superscript indicates the matrix inverse function.

We employ Sharpe's [1963] well-known market model, a single-index risk model for security returns based on the return of the capitalization-weighted market portfolio, r_M. In Sharpe's market model, the returns on the ith security are assumed to follow $r_i = \alpha_i + \beta_i r_M + \varepsilon_i$ where ε_i is a zero-mean random variable with variance $\sigma_{\varepsilon i}^2$ that is uncorrelated with β_i or any other ε_j. Let σ_M^2 denote the variance of the market return or, more generally, a single risk factor around which securities are assumed to co-vary. Using matrix notation, the N-by-N security covariance matrix associated with the single-index model is

$$\Omega = \boldsymbol{\beta\beta}'\sigma_M^2 + \text{Diag}(\sigma_\varepsilon^2) \quad \text{(A-3)}$$

where β is an N-by-1 vector of β_i, and σ_ε is an N-by-1 vector of $\sigma_{\varepsilon i}$. Using the Matrix Inversion Lemma (see Woodbury [1949]), the inverse covariance matrix is analytically solvable by

$$\Omega^{-1} = \text{Diag}(1/\sigma_\varepsilon^2) - \frac{(\boldsymbol{\beta}/\sigma_\varepsilon^2)(\boldsymbol{\beta}/\sigma_\varepsilon^2)'}{\frac{1}{\sigma_M^2} + (\boldsymbol{\beta}/\sigma_\varepsilon^2)'\boldsymbol{\beta}} \quad \text{(A-4)}$$

where $\beta/\sigma_\varepsilon^2$ is an N-by-1 vector of idiosyncratic risk–adjusted betas, $\beta_i/\sigma_{\varepsilon i}^2$. The substitution of Equation (A-4) into the general solution for the minimum-variance portfolio weights in Equation (A-2) and substantial algebra, results in the simple expression for the individual security weights shown as Equation (1), where β_{LS} is a *long–short threshold* beta calculated by

$$\beta_{LS} = \frac{\frac{1}{\sigma_M^2} + \Sigma \frac{\beta_i^2}{\sigma_{\varepsilon i}^2}}{\Sigma \frac{\beta_i}{\sigma_{\varepsilon i}^2}} \quad \text{(A-5)}$$

We refer to β_{LS} in Equation (A-5) as the long–short threshold beta because the value delineates positive and negative security weights in an unconstrained portfolio optimization. Under some simplifying assumptions (i.e., β_i and $\sigma_{\varepsilon i}$ are cross-sectionally uncorrelated and N is large), the value of β_{LS} is approximately equal to the mean beta (e.g., 1.0) plus the cross-sectional variance in betas (e.g., 0.04 for a cross-sectional ex ante beta standard deviation of 0.2). Equation (1) states that securities with $\beta_i < \beta_{LS}$ (i.e., a little more than half of the securities if $\beta_{LS} = 1.04$) have positive weights and that securities with $\beta_i > \beta_{LS}$ have negative weights.

A more novel result is that the form of Equation (1) is preserved for long-only constrained minimum-variance portfolios where β_L is the long-only threshold beta, calculated as in Equation (A-5), but only using securities in the constrained solution

$$\beta_L = \frac{\frac{1}{\sigma_M^2} + \Sigma_{\beta_i<\beta_L} \frac{\beta_i^2}{\sigma_{\varepsilon i}^2}}{\Sigma_{\beta_i<\beta_L} \frac{\beta_i}{\sigma_{\varepsilon i}^2}} \quad \text{(A-6)}$$

As a semi-formal proof of the long-only result in Equation (2), suppose that the value of β_L was already established and the investable set restricted to securities with beta values below that threshold. Then the original unconstrained optimization math in Equation (1) would hold and dictate the positive individual security weights. Now suppose a security with a beta above the threshold value were introduced into the investable set. If the negative optimal weight specified by Equation (1) were allowed, then the security would be included as a short position in the portfolio and would abide by the general optimality condition that the first derivative of portfolio risk with respect to a small change in any individual security weight is zero. We know, however, that the first derivative is positive at the constrained weight value of zero because second derivatives in a minimization are everywhere positive. A decrease in weight below zero is prohibited by the long-only constraint and an increase in weight above zero would increase portfolio risk, so the optimal weight on all securities with higher betas than the threshold value is fixed at zero.

As a practical matter, we verified that the constrained optimization solution specified in Equation (2) exactly matches a general numerical search routine for every security in all 504 monthly optimizations of this study. Note that Equation (A-6) is not technically a closed-form solution; the value of β_L is required for the conditional sums in the right side of the equation. But optimal weights can be calculated without numerical search routines simply by sorting securities from low to high ex ante beta and examining the running sums. Typical values for β_L are one standard deviation below the cross-sectional mean (i.e., about 0.8) so that less than 20% of the securities (those with the lowest betas) come into solution in long-only optimizations.

Employing Equation (1) along with the definition of a portfolio beta as the weighted average beta of the individual securities gives the unconstrained optimal portfolio's beta as

$$\beta_{MV} = \frac{\sigma_{MV}^2}{\sigma_M^2} \frac{1}{\beta_{LS}} \quad \text{(A-7)}$$

By extension (using the intuition of optimization on an investable subset), a similar expression holds for the long-only constrained portfolio beta, β_{LMV}, using the parameters σ_{LMV}^2 and β_L. Combining this result with the definitional decomposition of total portfolio risk into benchmark exposure risk and residual risk, $\sigma_P^2 = \beta_P^2\sigma_M^2 + \sigma_\varepsilon^2$, gives the ratio of portfolio to threshold beta as shown in Equation (3).

ENDNOTES

[1]The January 1968 start date is dictated by the availability of 60 months of historical returns for at least 1,000 stocks in the CRSP database. Exhibit 1 is similar to Exhibit 4 (Clarke, de Silva, and Thorley [2006]), but is extended through the end of the CRSP database available at the time of this study.

[2]We designate the regression intercept term that adjusts for the non-unitary beta as alpha, but note that "alpha" is often calculated as the simple difference between managed and benchmark portfolio returns (i.e., the managed portfolio beta is assumed to be one).

[3] The 60-month historical variance and covariance terms are calculated without subtracting in-sample means (i.e., average squared returns and return cross-products) although this has little impact on the empirical results. In other words, the 1,000-by-1,000 covariance matrix is simply the product of the 1,000-by-60 historical excess return matrix multiplied by its transpose.

[4] For simplicity and ease of replication, we use a constant shrinkage factor of 50% over time. The exact shrinkage factor for the two-parameter model specified by Ledoit and Wolf [2004] varies from month to month with an average value of about 54% (i.e., individual covariance matrix elements are shifted 54% towards their cross-sectional mean value).

[5] We do not impose size limits on individual security weights, limit industry exposures, or impose tracking error constraints. We use the SAS Procedure NLP for numerical optimization, but selected months were cross-verified using alternative software routines with identical results.

[6] Using the matrix notation defined in the technical appendix, the formulas for deriving single index model parameters from the sample covariance matrix, $\mathbf{\Omega}$, are $\sigma_M^2 = \mathbf{w}'_M \,\mathbf{\Omega}\, \mathbf{w}_M$, $\beta = \mathbf{\Omega}\, \mathbf{w}_M / \sigma_M^2$ and $\sigma_\varepsilon^2 = \text{Diag}(\Omega) - \beta^2 \sigma_M^2$, where $\mathbf{w}_M$ is a vector of security weights for the market portfolio.

[7] Specifically, we conducted 1,000-observation cross-sectional regressions in selected months of 12-month realized betas on 60-month historical betas and found the estimated coefficient of these regressions is about 0.50. A similar rule of thumb embedded in on-line security beta estimates shrinks historical values by one-third rather than by one-half towards the theoretical mean value of one. Other beta shrinkage procedures based on different estimation windows and investable sets are provided by Vasicek [1973] and Blume [1975].

[8] The statistics are from MCSI and are based on the Barra Global Equity Model (GEM2) for December 2009.

[9] The Fama–French-like volatile-minus-stable (VMS) idiosyncratic volatility factor in the study Clarke, de Silva, and Thorley [2010] has a −0.88% annualized return from 1968 to 2009, also the time period of this study. The return on a similarly constructed Fama–French-like beta factor is quite similar at −0.83%, and the time-series correlation coefficient between VMS returns and the beta-factor returns is 0.921.

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